Weak singularities in large-scale structure: identification and workaround

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Overview

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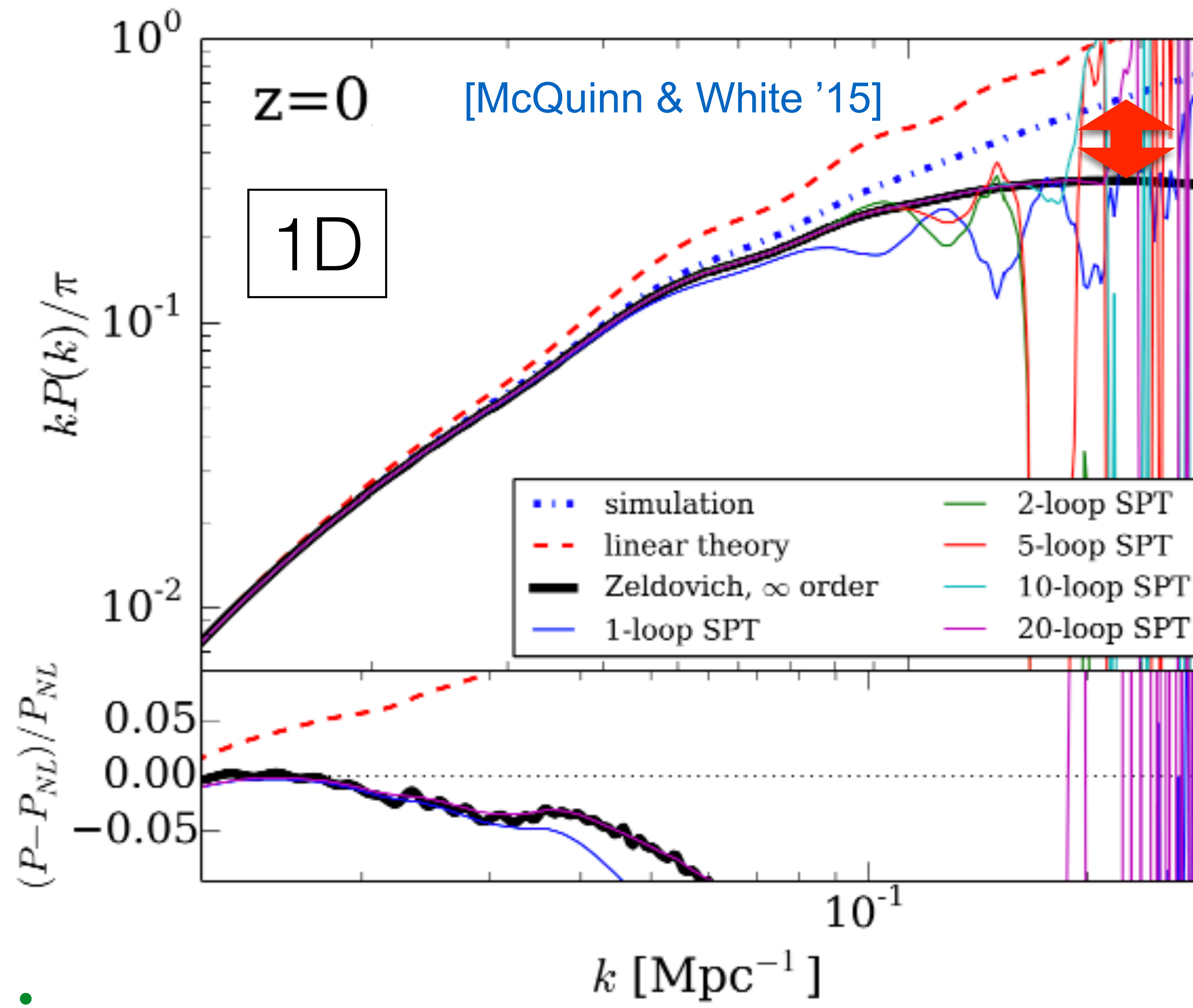
- At early times in structure formation, cold dark matter (CDM) is in the **single-stream regime** that comes with a single-valued velocity
- **Collisionless** nature of CDM leads to crossing of trajectories, called shell-crossing (where the density $\delta = (\rho - \bar{\rho})/\bar{\rho} \to \infty$)
- focus today: use suitable initial conditions to follow **analytically** trajectories into the **multi-stream regime**
- in the multi-stream regime, particle trajectories exhibit weakly singular behaviour (e.g. **kinks in the acceleration field**)
- confirmed by high-resolution N-body simulations
- Lastly, a semiclassical approach that is free of singularities

Motivation



Shell-crossing / multi-streaming effects are key **theoretical uncertainties** for the matter power spectrum

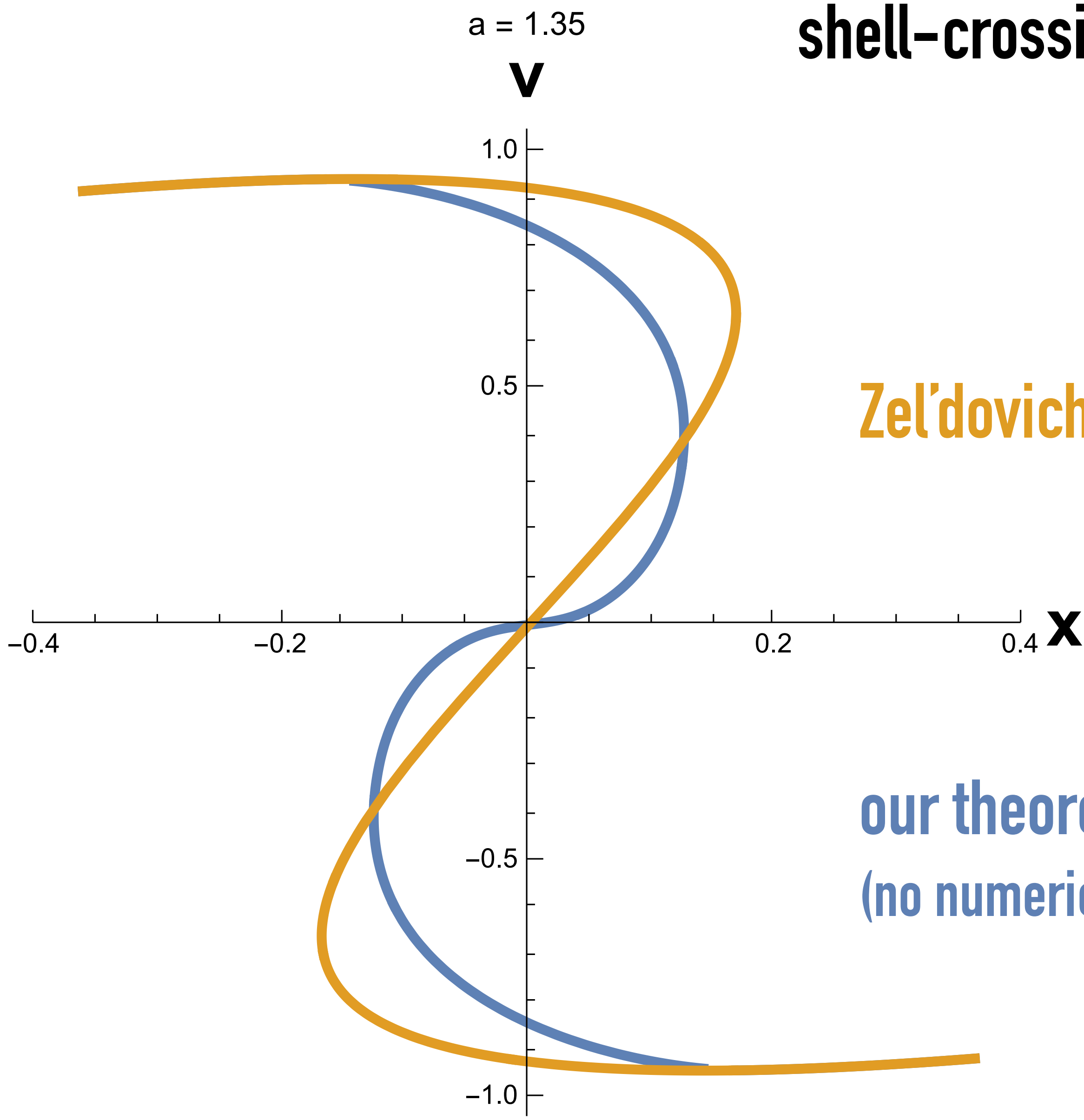
Analytical insight could assist in…

- closing the gap between theory and numerics
- make numerical simulations more efficient (including fastPM, COLA)
- gather information on counter / UV terms for effective theories

Sneak Preview: phase-space evolution in 1D

[CR, Hahn & Frisch, in prep.]

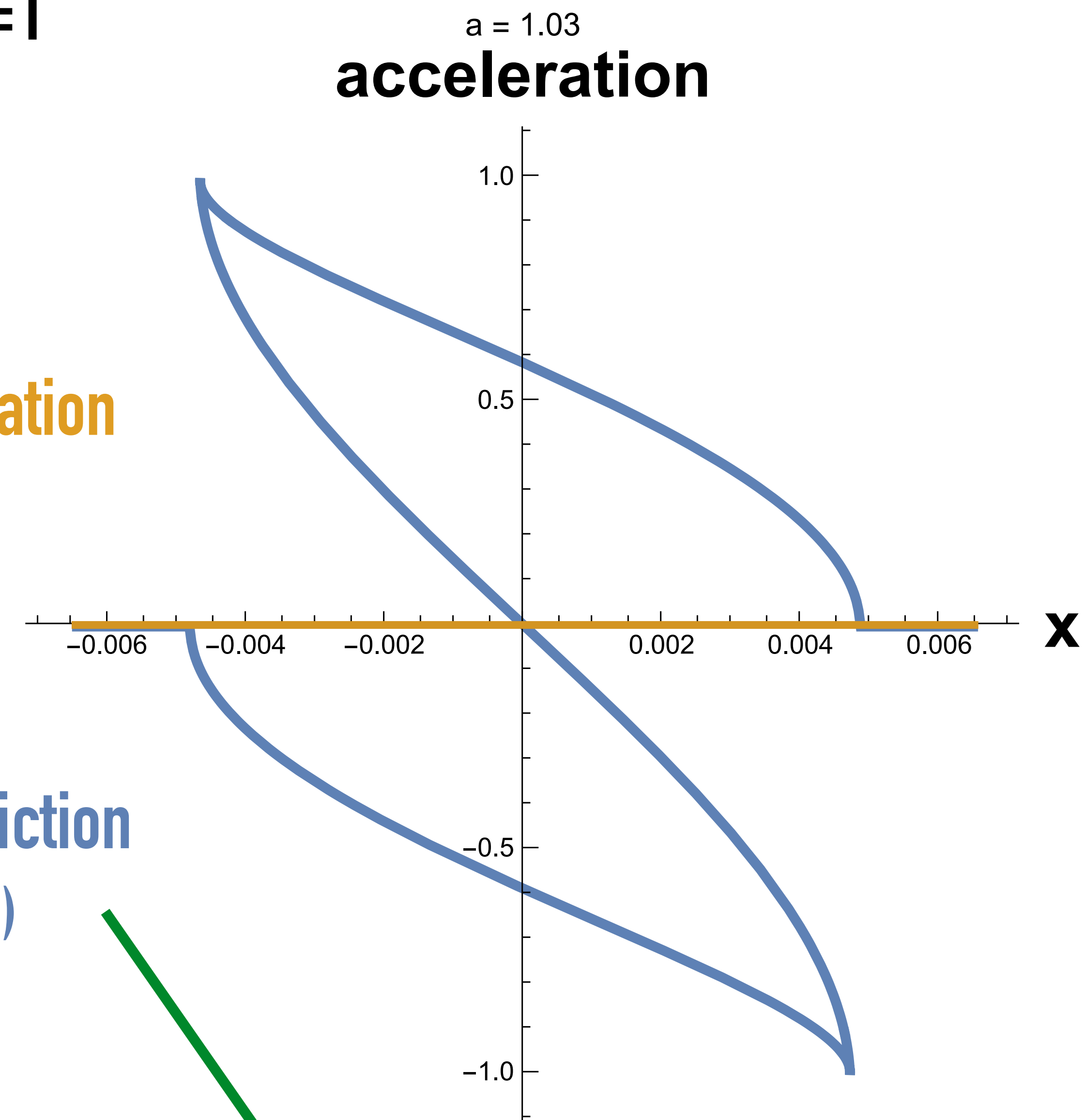
shell-crossing is at a=1



Zel'dovich approximation

our theoretical prediction

(no numerics required!!)



agrees with N-body simulations to very high precision!

acceleration is locally not differentiable -> weak singularities in Vlasov-Poisson

Single-stream regime

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- First solve for the **single-stream regime** since **initial conditions** for the **multi-stream regime** are set at **shell-crossing**

- **central** object: the Lagrangian map $\boldsymbol{q} \mapsto \boldsymbol{x}(\boldsymbol{q}, a) = \boldsymbol{q} + \boldsymbol{\xi}(\boldsymbol{q}, a)$

  initial position of fluid particles ($\boldsymbol{q}$); current position of fluid particles ($\boldsymbol{x}$); displacement field ($\boldsymbol{\xi}$)

- perturbative framework: Lagrangian perturbation theory (LPT)

  where the displacement is expanded in a Taylor series

- **Exact analytical solutions** in the single-stream regime for

  (= representable by converging Taylor series in LPT)

  - 1D collapse [Novikov ’69, Zel’dovich ’69]
  - quasi-1D collapse [CR & Frisch ’17]
  - [spherical collapse (top hat)] excluding shell-crossing; see later [Peebles ‘67]
  - quasi-spherical collapse (perturbed top hat) [CR ’17]

1D vs. quasi-1D initial data in 3D

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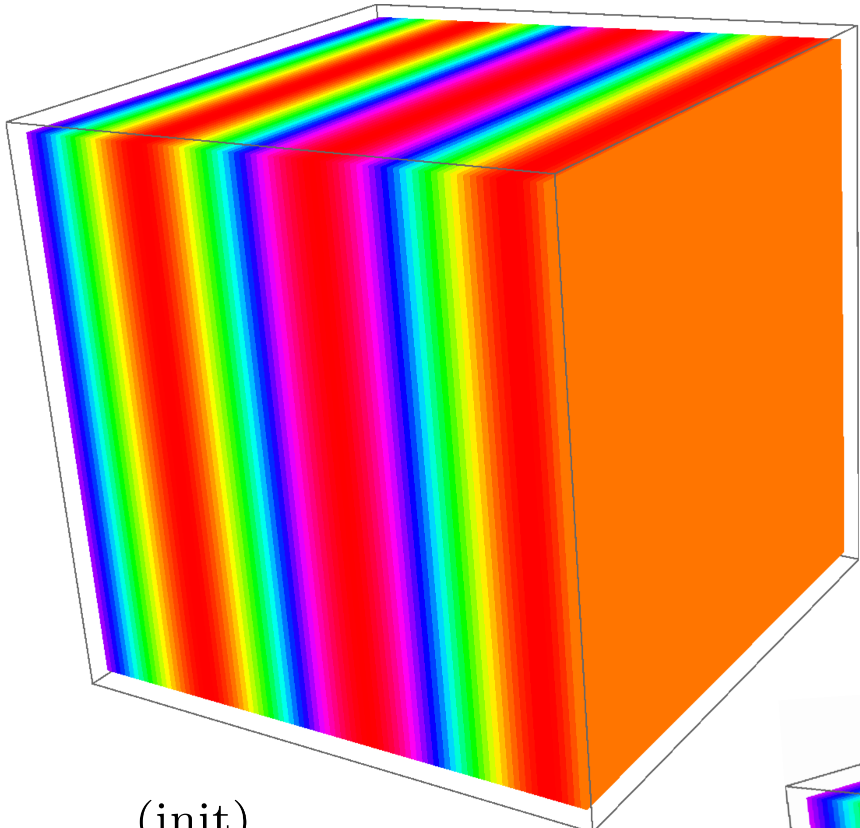
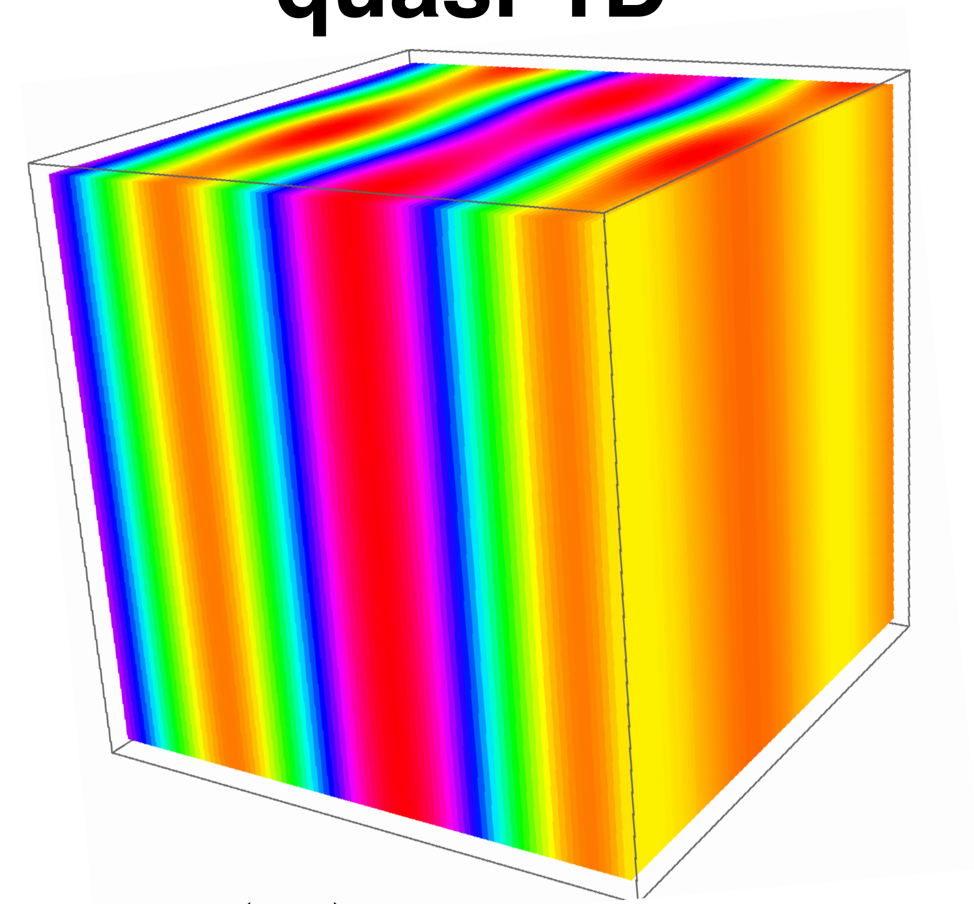
1D



$\varphi_{\rm g}^{\rm (init)} = \cos q_1$

examples

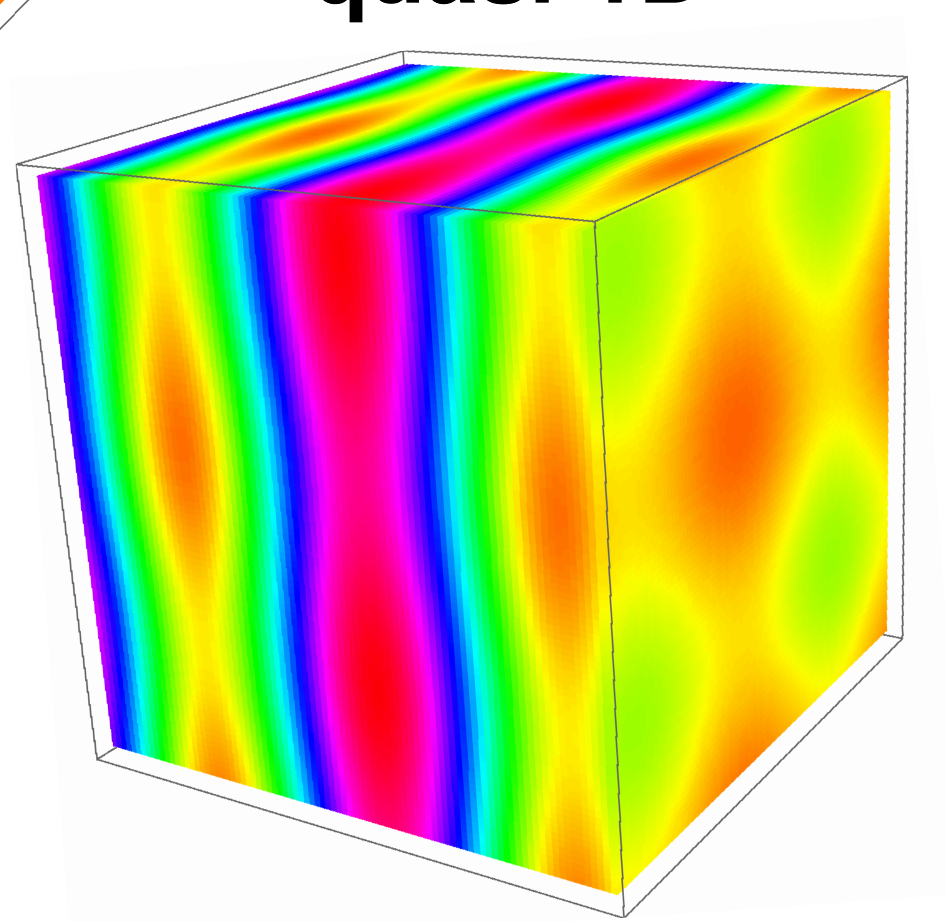
quasi-1D



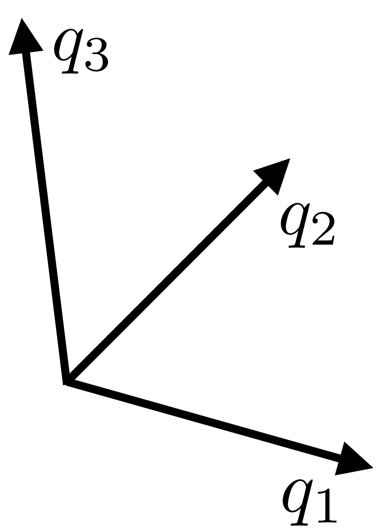
$\varphi_{\rm g}^{\rm (init)} = \cos q_1 + 0.1 \sin q_2$

initial gravitational potential

quasi-1D



$\varphi_{\rm g}^{\rm (init)} = \cos q_1 + 0.1 \sin q_2 + 0.1 \sin q_3$



LPT solutions at shell-crossing

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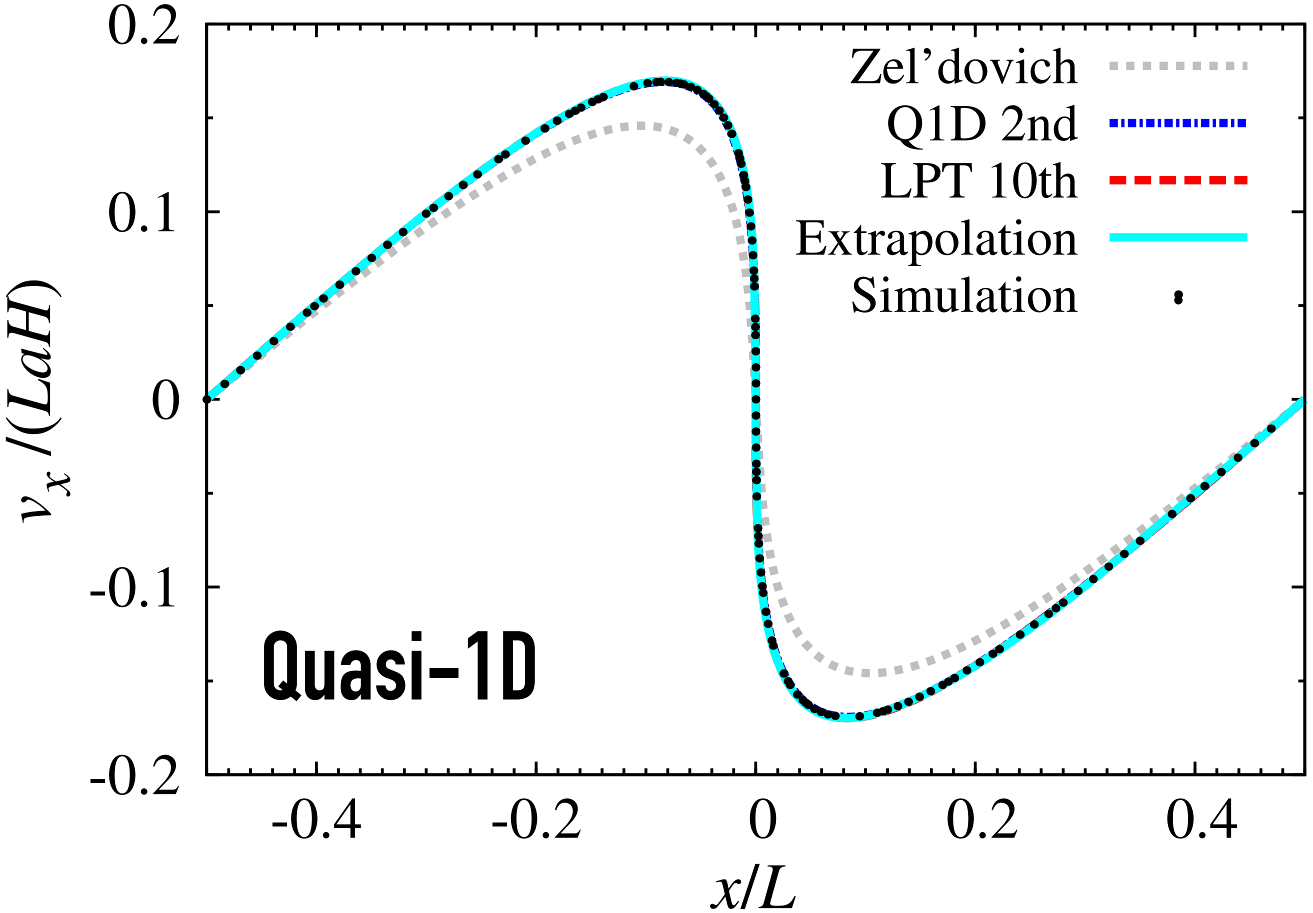
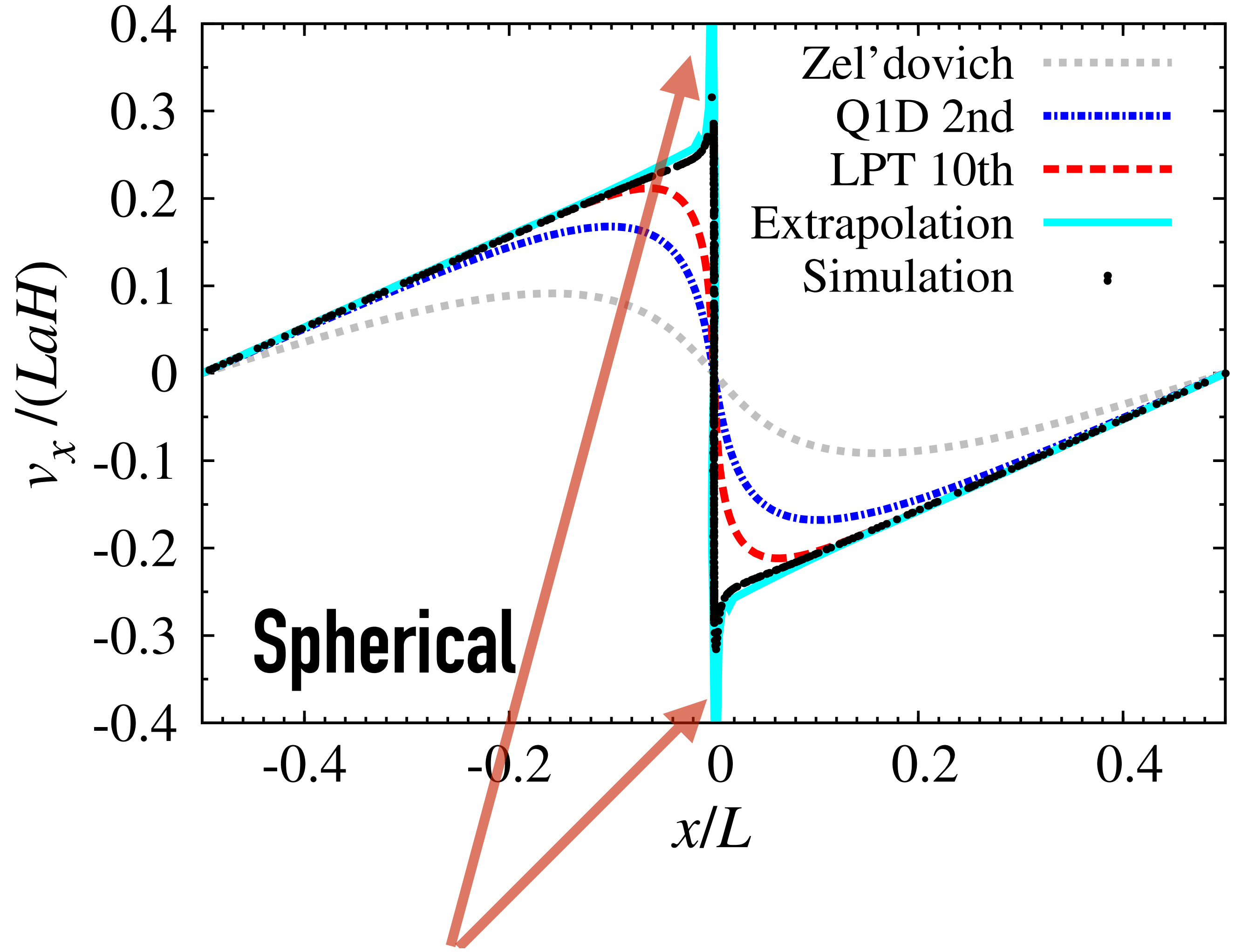
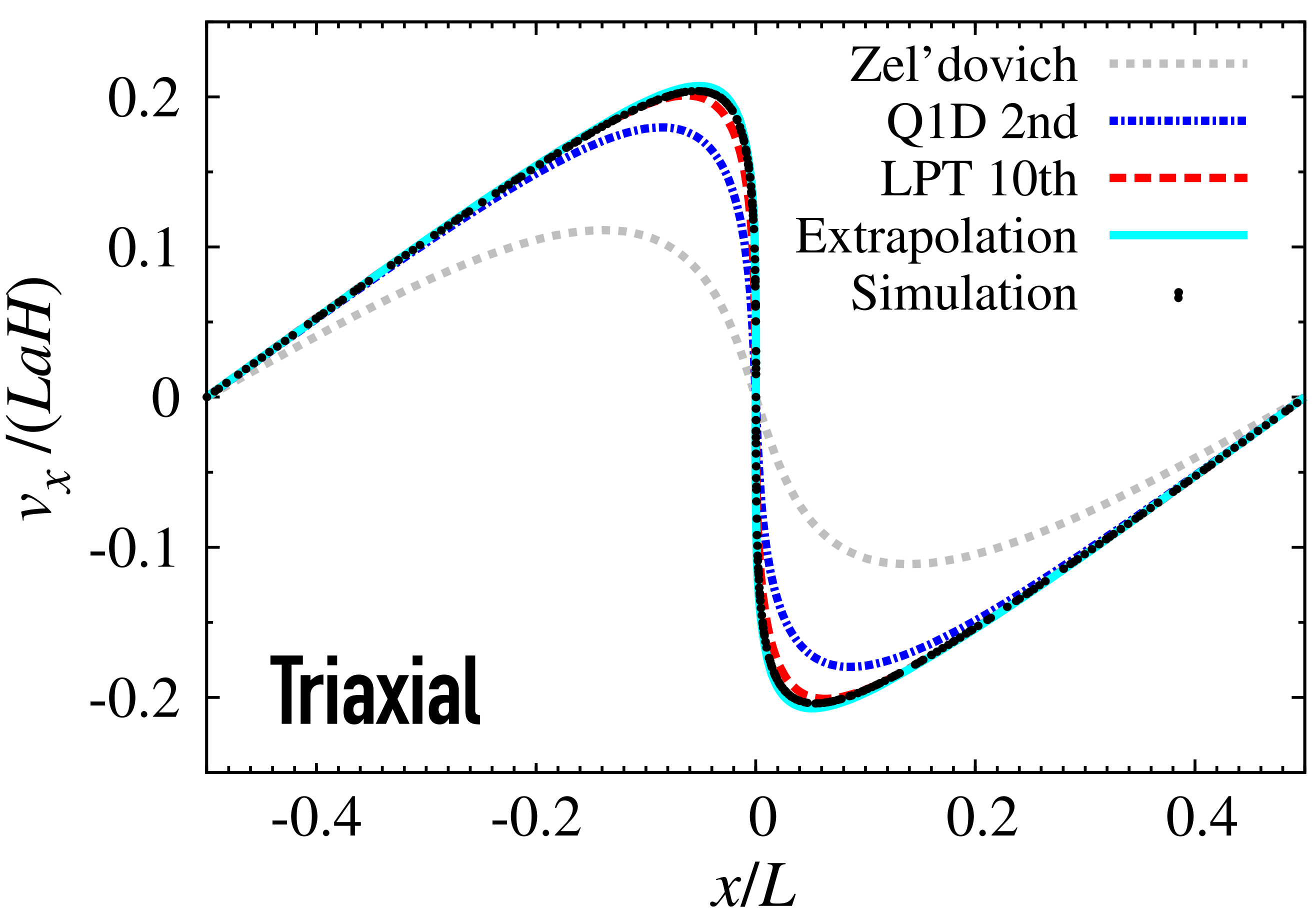
[Saga, Taruya & Colombi, PRL 2018]







singular velocity at shell-crossing

"LPT" is Lagrangian perturbation theory
"Q1D" the adapted PT for quasi-1D

Convergence for generic ICs

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- Except for singularities in the spherical collapse (which are removed in the quasi-spherical collapse), **LPT is converging, probably** even until shell-crossing

- For **cosmological ICs**, only a **lower bound on the radius** of convergence is known

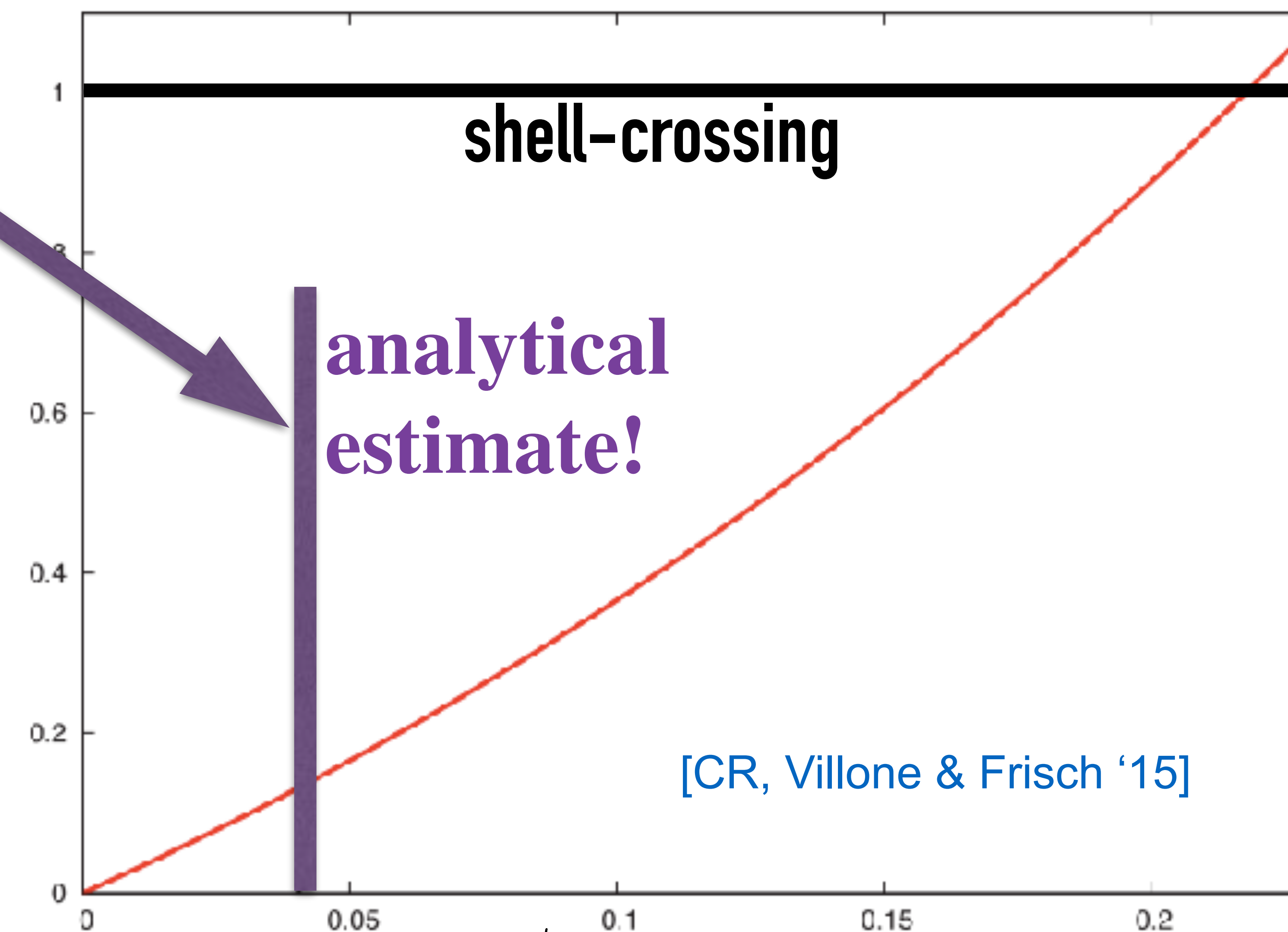
  [Zheligovsky & Frisch '14]

- Numerical studies of radius of convergence are needed

- If shell-crossing cannot be reached in a single time-step: analytic continuation!

  [both in preparation]

$|\delta/(1+\delta)| \equiv |J-1|$



ζ (generating function of displacement)

From single to multi-stream

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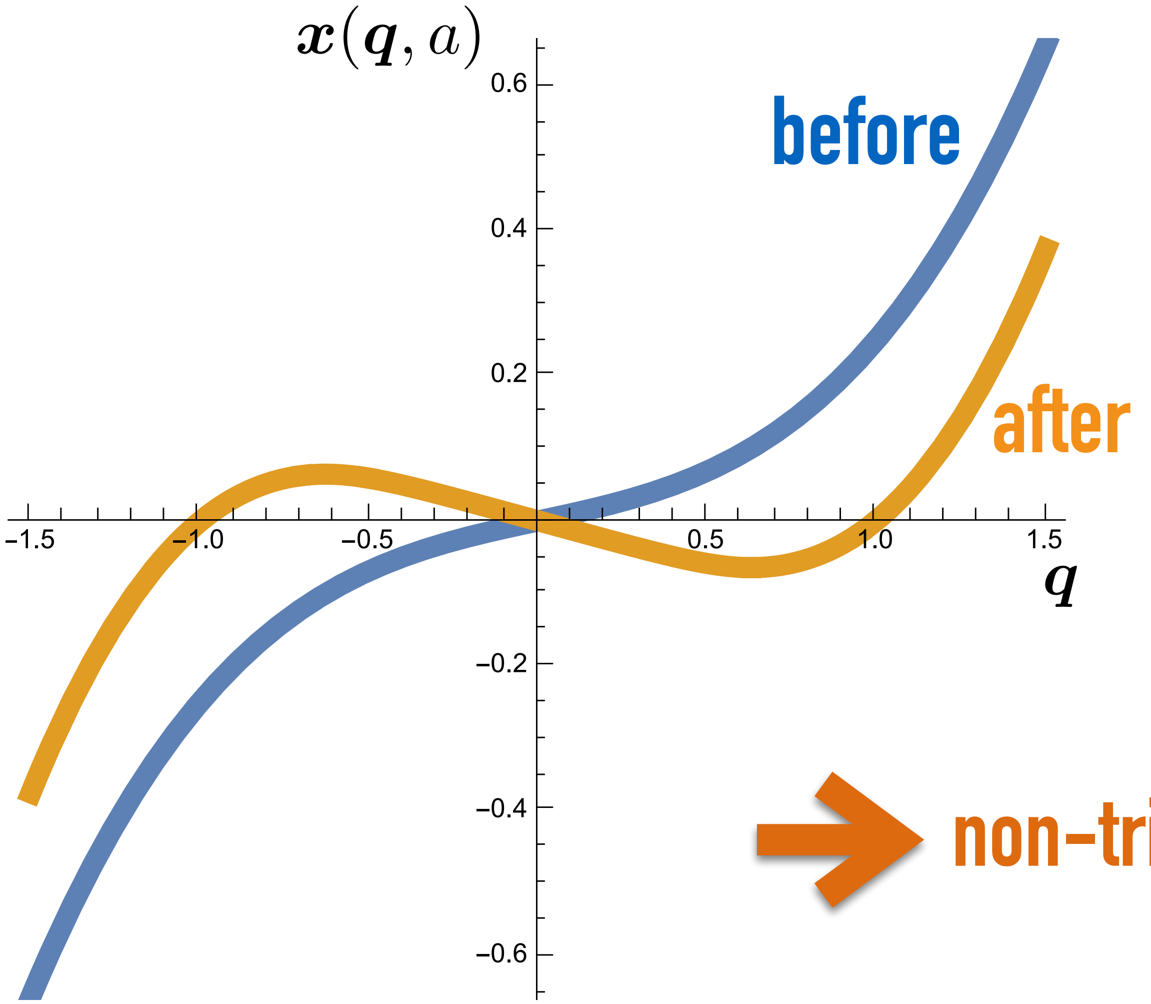
In Lagrangian coordinates, momentum conservation is (ignoring Hubble)

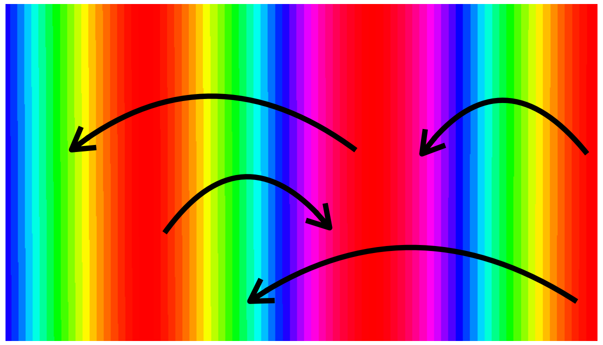
$$\ddot{\boldsymbol{x}}(\boldsymbol{q},a) \propto -\underbrace{\nabla_x \varphi_{\rm g}(\boldsymbol{x}(\boldsymbol{q},a))}_{\text{gravitational force}}$$

$\left(\nabla_x^2 \varphi_{\rm g} \propto \delta \right)$



Before shell-crossing, for each $\boldsymbol{q}$-particle there is one final position $\boldsymbol{x}$. **After the first shell-crossing** up to 3 different $\boldsymbol{q}$'s point to the same position. (Etc.)

→ non-trivial mass reshuffling (here 1D):



Computation of the gravitational force is non-trivial after shell-crossing. Besides that momentum conservation is unchanged.

Multi-stream computation

- Until shell-crossing, the gravitational force is given by ZA (in 1D).
  To leading order, this also holds shortly after (due to momentum conservation)

$$\ddot{\boldsymbol{x}}_{\rm PZA}(\boldsymbol{q},a) \propto -\nabla_x \varphi_{\rm g}(\boldsymbol{x}_{\rm ZA}(\boldsymbol{q},a)) \tag{1}$$

- To determine the force, Taruya & Colombi '17 use a (non-local) Green's function approach; some integrals need to be approximated

- To exploit the nonlinear power of LPT, we use the local expression

$$\begin{aligned} \delta(\boldsymbol{x}(\boldsymbol{q},a)) &= \int \delta_{\rm D}^{(3)}\left[\boldsymbol{x}(\boldsymbol{q},a) - \boldsymbol{x}(\boldsymbol{q}',a)\right] {\rm d}^3 q' - 1 \\ &= \int \sum_{k=1}^{n} \frac{\delta_{\rm D}^{(3)}\left[\boldsymbol{q}' - \boldsymbol{q}_n\right]}{|\det[\nabla_q \boldsymbol{x}(\boldsymbol{q}_n,a)]|} {\rm d}^3 q' - 1 \end{aligned}$$

- Analytical solutions for (1) are no simple power laws [see also: Pietroni '18]

Results in 1D

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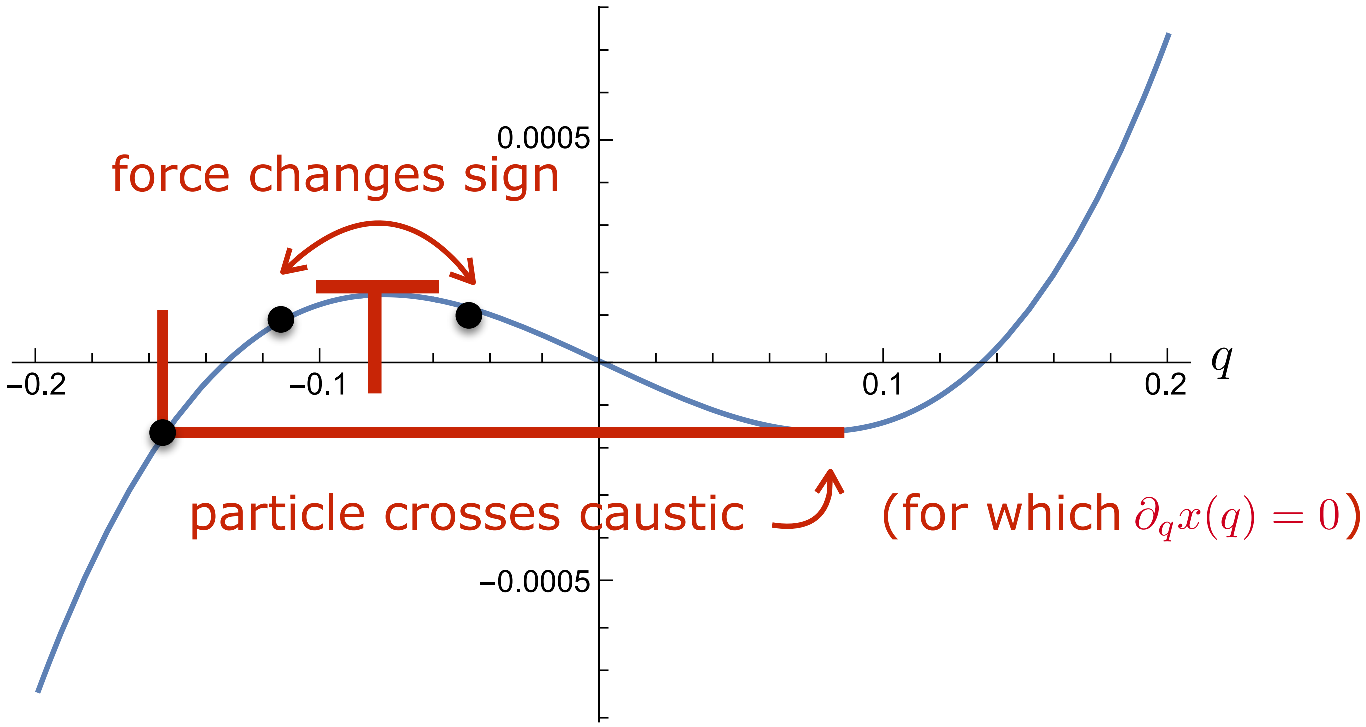
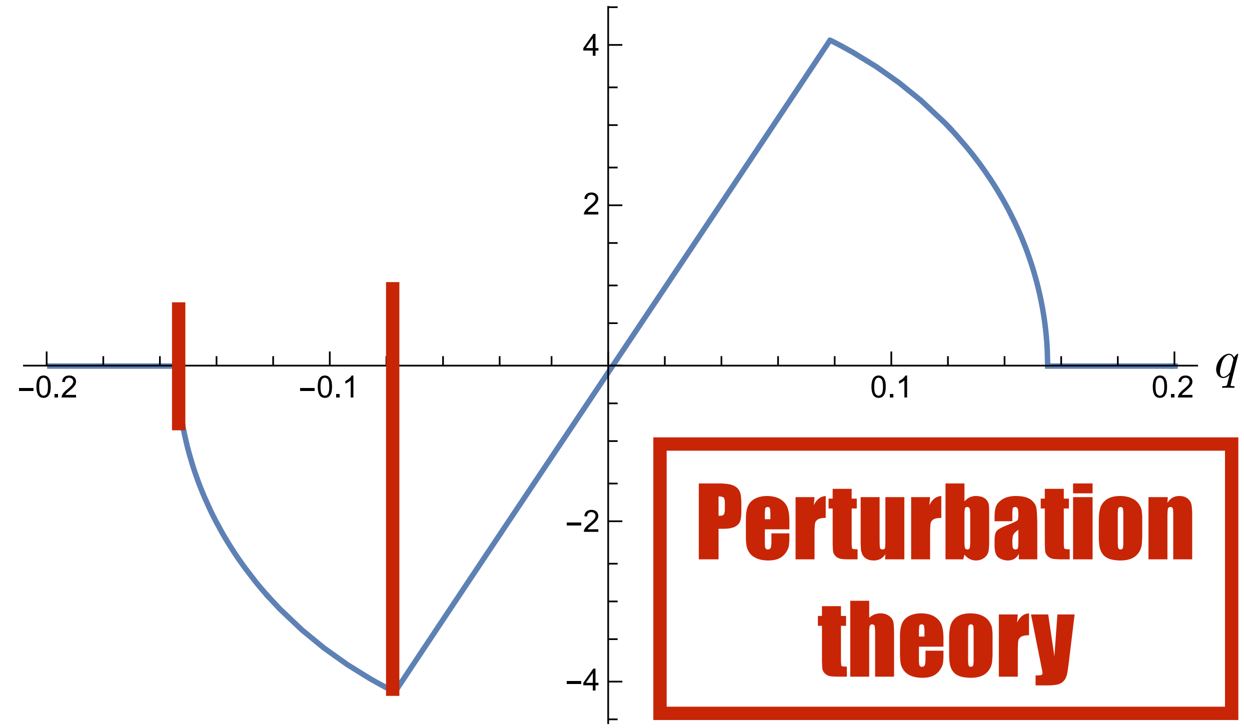
[CR, Frisch & Hahn, in prep.]

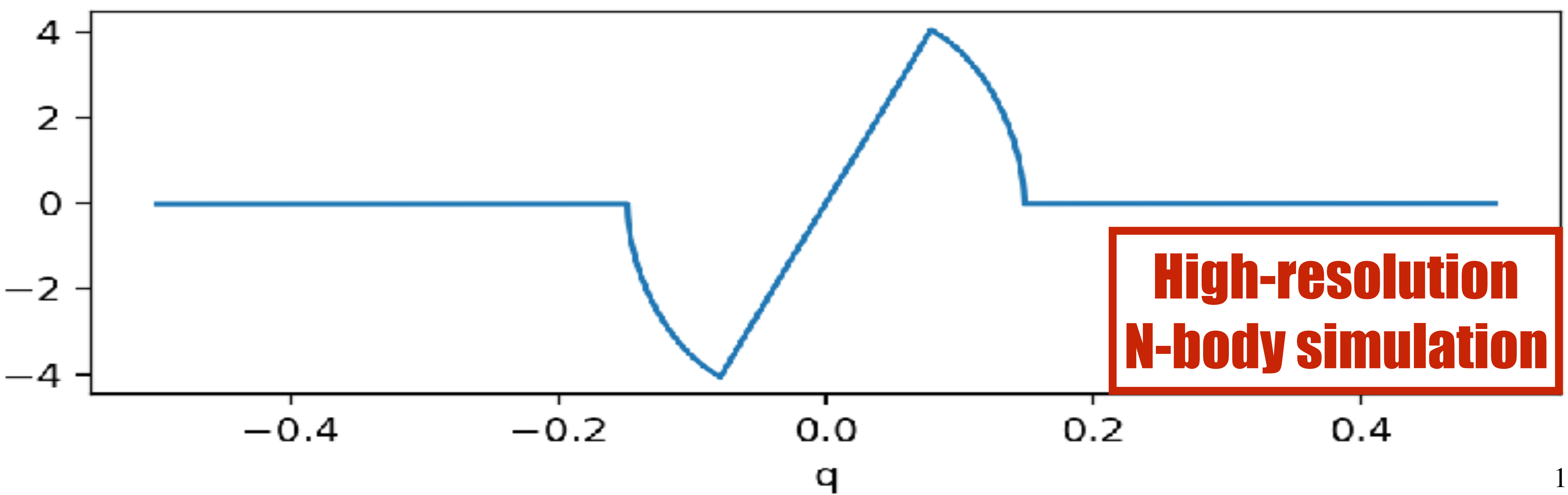
post-collapse particle trajectory $x(q)$



post-collapse acceleration $\ddot{x}(q)$



post-collapse acceleration $\ddot{x}(q)$



Regulating singular behaviour

Singular behaviour can be handled by high-precision methods. Alternatively, **singular features** may be **regulated** by

- **deviating from perfect coldness**:
  but then never pure single-stream, theoretical modelling complicated, requires in general full-fledged Vlasov-Poisson

- **employing semiclassical (Schrödinger-like) descriptions:**
  - correspondence: Schrödinger → Vlasov for $\hbar \to 0$
  - $\hbar > 0$ acts as a softening scale that regulates singularities
  - Widrow & Kaiser ’93 introduced Schrödinger-Poisson for LSS
  - we build on the related but different approach of Short & Coles ’06 which we motivate in the following from a QFT perspective

[Uhlemann, CR, Gosenca & Hahn ’18]

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[Uhlemann, CR, Gosenca & Hahn ’18]

- **propagator** for the Zel’dovich approximation: $K_{\rm ZA} \propto \exp\left\{ \frac{\rm i}{\hbar} S_{\rm ZA} \right\}$
with the ZA action

$$S_{\rm ZA} = \underbrace{\frac{1}{2}(\boldsymbol{x}-\boldsymbol{q})}_{\text{particle displacement}} \cdot \underbrace{\frac{\boldsymbol{x}-\boldsymbol{q}}{a}}_{\text{velocity}}$$

particle displacement times **velocity**

$\boldsymbol{q}$ initial position at $a=0$
$\boldsymbol{x}$ final position at a

- $K_{\rm ZA}$ propagates a wave function from initial to final state

$$\psi_{\rm ZA}(\boldsymbol{x};a) = \int {\rm d}^3 q\, K_{\rm ZA}(\boldsymbol{x},\boldsymbol{q};a)\, \psi_{\rm ZA}^{\rm (ini)}(\boldsymbol{q})$$

- governed by potential-free Schrödinger-type equations

$${\rm i}\hbar\partial_a K_{\rm ZA} = -\frac{\hbar^2}{2}\nabla_x^2 K_{\rm ZA}\,, \qquad {\rm i}\hbar\partial_a \psi_{\rm ZA} = -\frac{\hbar^2}{2}\nabla_x^2 \psi_{\rm ZA}$$

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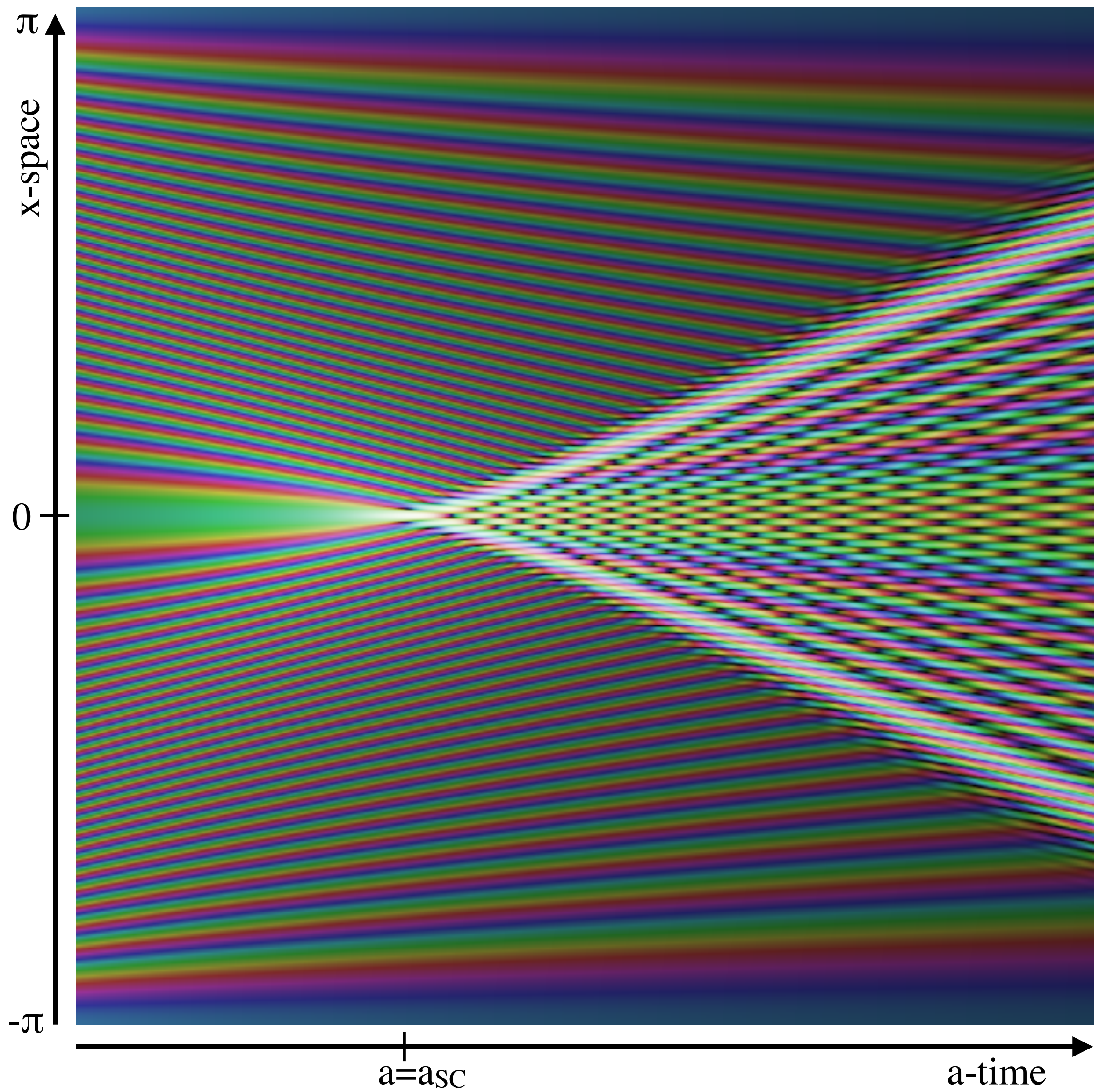
[Uhlemann, CR, Gosenca & Hahn ’18]

Solution for the wave function in 1D ($\hbar = 0.01$):



brightness is amplitude $\propto \sqrt{1+\delta}$, color the phase of ψ_{ZA}

- ✓ free of singularities for $\hbar > 0$
- ✓ some built-in multi-streaming, appearing as interference pattern
- ✓ agrees excellently with classical ZA before shell-crossing
- ✗ no full-fledged multi-streaming (no interaction potential)
- ✗ accurate only for 1D ICs (doesn’t include tidal interactions)

[Uhlemann, CR, Gosenca & Hahn '18]

- Let's include an effective potential $V_{\rm eff}$

$$\mathrm{i}\hbar\partial_a K = -\frac{\hbar^2}{2}\nabla_x^2 K + V_{\rm eff} K \,, \qquad \mathrm{i}\hbar\partial_a \psi = -\frac{\hbar^2}{2}\nabla_x^2 \psi + V_{\rm eff} \psi \,,$$

and set for the propagator $K \propto \exp\left\{ \frac{\mathrm{i}}{\hbar} \left[S_{\rm ZA} + S_{\rm int} \right] \right\}$.

- Perturbative solutions for $S_{\rm int}$ and thus for K can be obtained, provided one has a model for $V_{\rm eff}$

- In Uhlemann, CR++ '18 we focus on **gravitational tidal interactions**. Full-fledged multi-streaming will be done in a follow-up work.

- To include tidal interactions, we determine $V_{\rm eff}$ by using standard cosmological perturbation theory

(up to second order, one can equivalently also solve for $V_{\rm eff}$ in the Madelung representation $\psi = \sqrt{1+\delta}\,\exp(-\mathrm{i}\phi/\hbar)$)

[Uhlemann, CR, Gosenca & Hahn '18]

- In the **classical limit** $\hbar \to 0$, we recover results from Lagrangian perturbation theory (up to second order, "2LPT"), even without introducing the Lagrangian map!

- For $\hbar > 0$, the **propagator** has the **operational structure** of a numerical kick-drift-kick scheme, which simplifies the implementation greatly

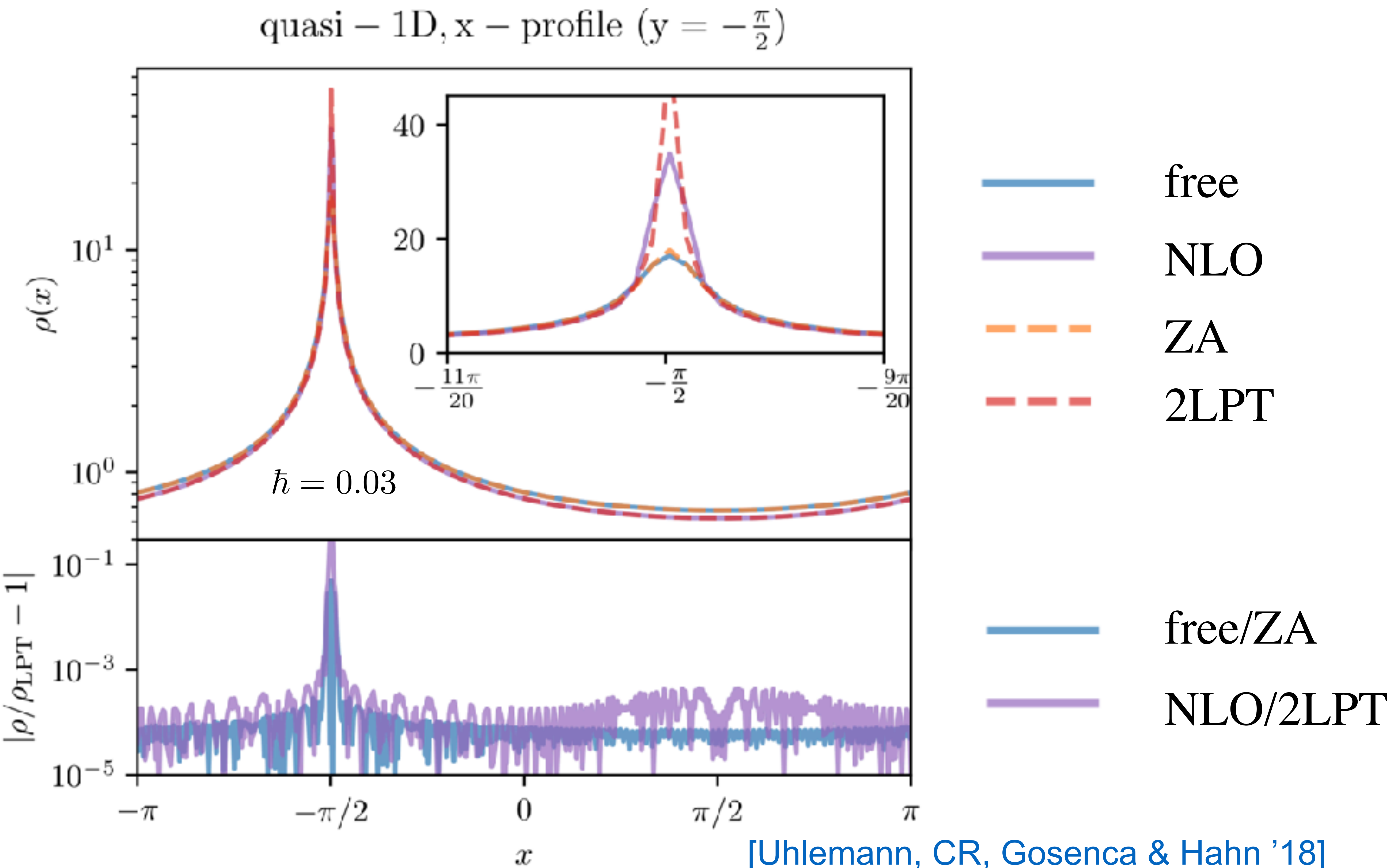
$$\textit{one time-step} = \underbrace{\exp\left(\frac{\epsilon\hat{V}}{2}\right)}_{\text{kick}} \underbrace{\exp\left(\epsilon\hat{T}\right)}_{\text{drift}} \underbrace{\exp\left(\frac{\epsilon\hat{V}}{2}\right)}_{\text{kick}}$$

$\epsilon = -i\hbar a$

cf. Baker-Campbell-Hausdorff formula

Results beyond 1D

Density $1+\delta = |\psi|^2$ **shortly** before shell-crossing:



[Uhlemann, CR, Gosenca & Hahn '18]

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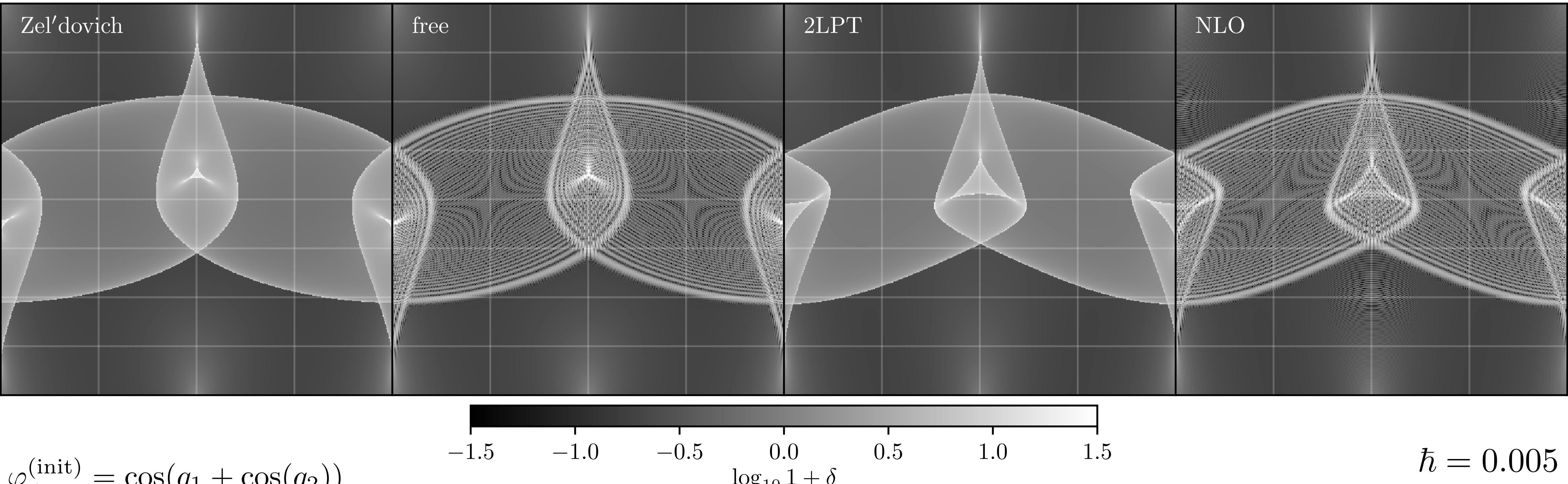
[Uhlemann, CR, Gosenca & Hahn ’18]

Density $1 + \delta = |\psi|^2$ for the phased wave problem in 2D



$\varphi_{\rm g}^{\rm (init)} = \cos(q_1 + \cos(q_2))$

$\hbar = 0.005$

Schrödinger solutions: directly obtained by evaluating the propagator on a mesh.

Zel’dovich and 2LPT solutions: require particle realizations from which the tessellated phase-space sheet can be constructed.

Vorticity through shell-crossing

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[Uhlemann, CR, Gosenca & Hahn ’18]

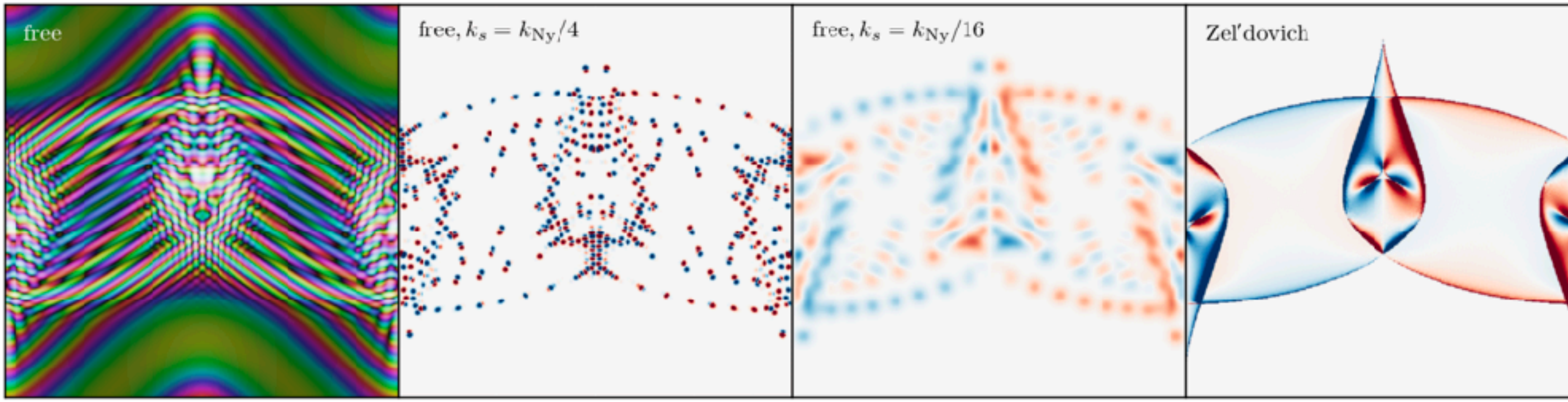

FIG. 5. The wave function ψ (left panel, shown using domain coloring), as well as the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times (\boldsymbol{j}/[1+\delta])$ (other panels) for the phased wave problem. The second and third panels from the left show the vorticity obtained using the free propagator, filtered with a Gaussian filter in Fourier space on scales of 1/4 and 1/16 the Nyquist wave number to highlight both the large-scale transversal modes and the topological defects from which they arise. The right-most panel shows the corresponding vorticity using the Zel'dovich approximation. Time and initial conditions are identical to Fig. 4, but in order to highlight the role of $\hbar$, it has been increased to $\hbar = 0.03$. The color scale for vorticity has been adjusted to highlight best the various features in each panel.

Conclusions & Outlook

- weak singularities in the multi-stream regime of structure formation, due to the perfect coldness of CDM
- we have significantly closed the gap between theory and numerics
- still a lot of work to do beyond 1D
- **singular features** may be **regulated** by
  - **deviating from perfect coldness**:
    no single-stream, theoretical modelling complicated,
    requires full-fledged Vlasov-Poisson
  - **employing semiclassical (Schrödinger-like) descriptions**
    still has to be generalized to include full-fledged multi-streaming