Cosmic Acceleration from Modified Gravity:

A Worked Example

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Why Study $f(R)$?

- Cosmic acceleration, like the cosmological constant, can either be viewed as arising from

  Missing, or dark energy, with $w \equiv \frac{\bar{p}}{\bar{\rho}} < -1/3$

  Modification of gravity on large scales

  $$G_{\mu\nu} = 8\pi G \left( T^{M}_{\mu\nu} + T^{DE}_{\mu\nu} \right)$$

  $$F(g_{\mu\nu}) + G_{\mu\nu} = 8\pi G T^{M}_{\mu\nu}$$

- Proof of principle models for both exist: quintessence, k-essence; DGP braneworld acceleration, $f(R)$ modified action

- Compelling models for either explanation lacking

- Study models as illustrative toy models whose features can be generalized
Three Regimes

• Three regimes defined by $\gamma = -\Phi/\Psi$ BUT with different dynamics

• Examples $f(R)$ and DGP braneworld acceleration

• Parameterized Post-Friedmann description

• Non-linear regime follows a halo paradigm but a full parameterization still lacking and theoretical, examples few: $f(R)$ now fully worked

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**General Relativistic Non-Linear Regime**

- $r_*$
  - halos, galaxy

**Scalar-Tensor Regime**

- $r_c$
  - large scale structure

**Conserved-Curvature Regime**

- $r$
  - CMB
Outline

- $f(R)$ Basics and Background
- Linear Theory Predictions
- N-body Simulations and the Chameleon

- Collaborators:
  - Marcos Lima
  - Hiro Oyaizu
  - Hiranya Peiris
  - Iggy Sawicki
  - Fabian Schmidt
  - Yong-Seon Song
$f(R)$ Basics
Cast of $f(R)$ Characters

- $R$: Ricci scalar or “curvature”
- $f(R)$: modified action (Starobinsky 1980; Carroll et al 2004)

$$S = \int d^4 x \sqrt{-g} \left[ \frac{R + f(R)}{16\pi G} + \mathcal{L}_m \right]$$
Cast of $f(R)$ Characters

- $R$: Ricci scalar or "curvature"
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$$S = \int d^4x \sqrt{-g} \left[ \frac{R + f(R)}{16\pi G} + \mathcal{L}_m \right]$$

- $f_R \equiv df/dR$: additional propagating scalar degree of freedom (metric variation)
- $f_{RR} \equiv d^2f/dR^2$: Compton wavelength of $f_R$ squared, inverse mass squared
- $B$: Compton wavelength of $f_R$ squared in units of the Hubble length

$$B \equiv \frac{f_{RR}}{1 + f_R} \frac{H}{H'}$$

- $' \equiv d/d\ln a$: scale factor as time coordinate
Modified Einstein Equation

- In the Jordan frame, gravity becomes 4th order but matter remains minimally coupled and separately conserved

\[ G_{\alpha\beta} + f_R R_{\alpha\beta} - \left( \frac{f}{2} - \square f_R \right) g_{\alpha\beta} - \nabla_\alpha \nabla_\beta f_R = 8\pi G T_{\alpha\beta} \]

- Trace can be interpreted as a scalar field equation for \( f_R \) with a density-dependent effective potential \( (\rho = 0) \)

\[ 3\square f_R + f_R R - 2 f = R - 8\pi G \rho \]
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\[ 3\Box f_R + f_R R - 2f = R - 8\pi G \rho \]

- For small deviations, \( |f_R| \ll 1 \) and \( |f/R| \ll 1 \),

\[ \Box f_R \approx \frac{1}{3} \left( R - 8\pi G \rho \right) \]

the field is sourced by the deviation from GR relation between curvature and density and has a mass

\[ m^2_{f_R} \approx \frac{1}{3} \frac{\partial R}{\partial f_R} = \frac{1}{3 f_{RR}} \]
Effective Potential

- Scalar $f_R$ rolls in an effective potential that depends on density.
- At high density, extrema is at GR $R=8\pi G\rho$.
- Minimum for $B>0$, pinning field to $|f_R| \ll 1$, maximum for $B<0$.

Sawicki & Hu (2007)
$f(R)$ Expansion History
Modified Friedmann Equation

- Expansion history parameterization: Friedmann equation becomes
  \[ H^2 - f_R(HH' + H^2) + \frac{1}{6} f + H^2 f_{RR} R' = \frac{8\pi G\rho}{3} \]

- Reverse engineering \( f(R) \) from the expansion history: for any desired \( H \), solve a 2nd order diffeq to find \( f(R) \)

- Allows a family of \( f(R) \) models, parameterized in terms of the Compton wavelength parameter \( B \)
Modified Friedmann Equation

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- Formally includes models where \( B < 0 \), such as \( f(R) = -\mu^4/R \), leading to confusion as to whether such models provide viable expansion histories

- Answer: no these have short-time scale tachyonic instabilities at high curvature and limit as \( B \to 0 \) from below is not GR

- \( B > 0 \) family has very different implications for structure formation but with identical distance-redshift relations
Expansion History Family of $f(R)$

- Each expansion history, matched by dark energy model $[w(z), \Omega_{DE}, H_0]$ corresponds to a family of $f(R)$ models due to its 4th order nature.
- Parameterized by $B \propto f_{RR} = \frac{d^2f}{dR^2}$ evaluated at $z=0$.

Song, Hu & Sawicki (2006)
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![Graph showing \(f(R)/R\) versus \(R/H_0^2\) for different values of \(B_0\).]
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![Graph showing the expansion history family of $f(R)$ models for different dark energy models.](image)

(a) $w=-1$, $\Omega_{DE}=0.76$

(b) $w=-0.9$, $\Omega_{DE}=0.73$

Song, Hu & Sawicki (2006)
Instability at High Curvature

- **Tachyonic instability** for negative mass squared $B<0$ makes **high curvature** regime increasingly **unstable**: high density ≠ high curvature
- Linear metric **perturbations** immediately drop the expansion history to low curvature solution

![Graph showing the relationship between curvature $R$ and scale factor $a$ for $B>0$ and $B<0$. The graph indicates the decrease in curvature with increasing scale factor for both $\Lambda CDM$ and $B>0$, $B<0$ cases.](image)

Sawicki & Hu (2007)
$f(R)$ Linear Theory
Three Regimes

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- Examples $f(R)$ and DGP braneworld acceleration
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General Relativistic Non-Linear Regime

Scalar-Tensor Regime

Conserved-Curvature Regime

$\ gamma_0$, galaxy $\ gamma_c$, large scale structure $\ gamma$ CMB
Curvature Conservation

- On superhorizon scales, energy momentum conservation and expansion history constrain the evolution of metric fluctuations (Bertschinger 2006)

- For adiabatic perturbations in a flat universe, conservation of comoving curvature applies $\zeta' = 0$ where $' \equiv d/d \ln a$ (Bardeen 1980)

- Gauge transformation to Newtonian gauge

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(1 + 2\Phi)dx^2$$

yields (Hu & Eisenstein 1999)

$$\Phi'' - \Psi' - \frac{H''}{H'}\Phi' - \left(\frac{H'}{H} - \frac{H''}{H'}\right)\Psi = 0$$

- Modified gravity theory supplies the closure relationship $\Phi = -\gamma(\ln a)\Psi$ between and expansion history $H = \dot{a}/a$ supplies rest.
Linear Theory for $f(R)$

- In $f(R)$ model, “superhorizon” behavior persists until Compton wavelength smaller than fluctuation wavelength $B^{1/2}(k/aH) < 1$
- Once Compton wavelength becomes larger than fluctuation

$$B^{1/2}(k/aH) > 1$$

perturbations are in scalar-tensor regime described by $\gamma = 1/2$. 
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- Small scale density growth enhanced and

$$8\pi G \rho > R$$

low curvature regime with order unity deviations from GR
- Transitions in the non-linear regime where the Compton wavelength can shrink via chameleon mechanism
- Given $k_{NL}/aH \gg 1$, even very small $f_R$ have scalar-tensor regime
Deviation Parameter

- Express the 4th order nature of equations as a deviation parameter

\[ \Phi'' - \Psi' - \frac{H''}{H'} \Phi' - \left( \frac{H'}{H} - \frac{H''}{H'} \right) \Psi = \left( \frac{k}{aH} \right)^2 B\epsilon \]

- Einstein equation become a second order equation for \( \epsilon \)
Deviation Parameter

- Express the 4th order nature of equations as a deviation parameter

\[
\Phi'' - \Psi' - \frac{H''}{H'} \Phi' - \left( \frac{H'}{H} - \frac{H''}{H'} \right) \Psi = \left( \frac{k}{aH} \right)^2 B \epsilon
\]

- Einstein equation become a second order equation for \( \epsilon \)

- In high redshift, high curvature \( R \) limit this is

\[
\epsilon'' + \left( \frac{7}{2} + 4 \frac{B'}{B} \right) \epsilon' + \frac{2}{B} \epsilon = \frac{1}{B} \times \text{metric sources}
\]

\[
B = \frac{f_{RR}}{1 + f_R} R' \frac{H}{H'}
\]

- \( R \to \infty, B \to 0 \) and for \( B < 0 \) short time-scale tachyonic instability appears making previous models not cosmologically viable

\[
f(R) = -M^{2+2n} / R^n
\]
Potential Growth

• On the stable $B>0$ branch, potential evolution reverses from decay to growth as wavelength becomes smaller than Compton scale

• Quasistatic equilibrium reached in linear theory with $\gamma=-\Phi/\Psi=1/2$ until non-linear effects restore $\gamma=1$
Integrated Sachs-Wolfe Effect

- CMB photons transit gravitational potentials of large-scale structure.
- If potential decays during transit, gravitational blueshift of infall is not cancelled by gravitational redshift of exit.
- Spatial curvature of gravitational potential leads to additional effect $\Delta T/T = -\Delta(\Phi-\Psi)$. 

Integrated Sachs-Wolfe Effect

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ISW Quadrupole

- **Reduction of large angle anisotropy** for $B_0 \sim 1$ for same expansion history and distances as $\Lambda$CDM
- **Well-tested small scale anisotropy unchanged**

$$(l(l+1)C_l^T)/2\pi \sim B_0 \sim 1$$

$$(l(l+1)C_l^T)/2\pi \sim 0 \text{ (CDM)}$$

$$(l(l+1)C_l^T)/2\pi \sim 1/2$$

$$(l(l+1)C_l^T)/2\pi \sim 3/2$$

**Song, Hu & Sawicki (2006)**
ISW-Galaxy Correlation

- Decaying potential: galaxy positions correlated with CMB
- Growing potential: galaxy positions anticorrelated with CMB
- Observations indicate correlation
Galaxy-ISW Anti-Correlation

- Large Compton wavelength $B^{1/2}$ creates potential growth which can anti-correlate galaxies and the CMB
- In tension with detections of positive correlations across a range of redshifts

Song, Peiris & Hu (2007)
Linear Power Spectrum

- Linear real space power spectrum enhanced on scales below Compton scale in the background
- Scale-dependent growth rate and potentially large deviations on small scales
$f(R)$ Non-Linear Evolution
Three Regimes

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**General Relativistic Non-Linear Regime**

$\mathbf{r}_* \quad \text{halos, galaxy}$

**Scalar-Tensor Regime**

$\mathbf{r}_c \quad \text{large scale structure}$

**Conserved-Curvature Regime**

$\mathbf{r} \quad \text{CMB}$
Non-Linear Chameleon

• For $f(R)$ the field equation

$$\nabla^2 f_R \approx \frac{1}{3} (\delta R(f_R) - 8\pi G\delta \rho)$$

is the non-linear equation that returns general relativity

• High curvature implies short Compton wavelength and suppressed deviations but requires a change in the field from the background value $\delta R(f_R)$

• Change in field is generated by density perturbations just like gravitational potential so that the chameleon appears only if

$$\Delta f_R \leq \frac{2}{3} \Phi,$$

else required field gradients too large despite $\delta R = 8\pi G\delta \rho$ being the local minimum of effective potential
Non-Linear Dynamics

- Supplement that with the **modified Poisson equation**

\[ \nabla^2 \Psi = \frac{16\pi G}{3} \delta \rho - \frac{1}{6} \delta R(f_R) \]

- Matter evolution given metric unchanged: usual *motion of matter* in a gravitational potential \( \Psi \)

- Prescription for *N*-body code

- **Particle Mesh** (PM) for the Poisson equation

- Field equation is a non-linear Poisson equation: *relaxation* method for \( f_R \)

- **Initial conditions** set to GR at high redshift
Environment Dependent Force

- Chameleon suppresses extra force (scalar field) in high density, deep potential regions

\( f_{R0} = 10^{-6} \)

\[ \text{density: } \max[\ln(1+\delta)] \quad \text{potential: } \min[\Psi] \quad \text{field: } \min[f_R/f_{R0}] \]
Environment Dependent Force

- For large background field, gradients in the scalar prevent the chameleon from appearing

• 512$^3$ PM-relaxation code resolves the chameleon transition to GR: greatly reduced non-linear effect

![N-body Power Spectrum](chart.png)

Artificially turning off the chameleon mechanism restores much of enhancement.
Models where the chameleon absent today (large field models) show residual effects from a high redshift chameleon
Distance Predicts Growth

- With smooth dark energy, distance predicts scale-invariant growth to a few percent - a falsifiable prediction

Scaling Relations

- Fitting functions based on normal gravity fail to capture chameleon and effect of extra forces on dark matter halos.

N-body Power Spectrum

- Models where the chameleon is absent today (large field models) show residual effects from a high redshift chameleon.
Mass Function

- Enhanced abundance of rare dark matter halos (clusters) with extra force

Halo Bias

- Halos at a fixed mass less rare and less highly biased

Halo Mass Correlation

- Enhanced forces vs lower bias

Halo Model

- Power spectrum trends also consistent with halos and modified collapse

$f(R)$ Solar System Tests
• **Density profile of Sun is not a constant density sphere - interior photosphere, chromosphere, corona**

• **Density drops by \(~25\) orders of magnitude - does curvature follow?**

Hu & Sawicki (2007)
Solar System Constraint

- **Cassini constraint on PPN** $|\gamma - 1| < 2.3 \times 10^{-5}$
- Easily satisfied if **galactic field is at potential minimum** $|f_{Rg}| < 4.9 \times 10^{-11}$
- Allows even **order unity** cosmological fields

Hu & Sawicki (2007)
Field Solution

- **Field solution smoothly relaxes from exterior value to high curvature interior value** $f_R \sim 0$, minimizing potential + kinetic

- **Juncture** is where thin-shell criterion is satisfied $|\Delta f_R| \sim \Delta \Phi$

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![Graph](image-url)

Hu & Sawicki (2007)

$n=4$
Solar Curvature

- Curvature drops suddenly as field moves slightly from zero
- Enters into low curvature regime where $R < 8\pi G\rho$

Hu & Sawicki (2007)
Galactic Thin Shell

- Galaxy must have a thin shell for interior to remain at high curvature
- Rotation curve $v/c \approx 10^{-3}$, $\Phi \approx 10^{-6} \sim |\Delta f_R|$ limits cosmological field
- Has the low cosmological curvature propagated through local group and galactic exterior?

Hu & Sawicki (2007)
Summary

- General lessons from $f(R)$ example – 3 regimes:
  - large scales: conservation determined
  - intermediate scales: scalar-tensor
  - small scales: GR in high density regions, modified in low
- Given fixed expansion history $f(R)$ has additional continuous parameter: Compton wavelength
- Enhanced gravitational forces below environment-dependent Compton scale affect growth of structure
- Enhancement hidden by non-linear chameleon mechanism at high curvature $\neq$ high density
- $N$-body (PM-relaxation) simulations show potentially observable differences in the power spectrum and mass function