Solving the Vlasov equation in 2 spatial dimensions with the Schrödinger method

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in collaboration with
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University of Cyprus

Fyzikální ústav
Akademie věd ČR, v. v. i.

1. **Introduction/Motivation**

Vlasov equation, Closing Boltzmann hierarchy

2. **Comparison** of

coarse grained Vlasov and
Schrödinger method (ScM) with

a) 2D simulations (*visually*)
b) General formulas and 1D simulations (*visually*)
c) 2D simulations (*quantitatively*)

3. **Discussion**

Vorticity, Effective dark matter equation of state

4. **Summary & Outlook**

Vlasov equation: Recap and definitions

Continuous phase space distribution function $f(t, \boldsymbol{x}, \boldsymbol{u})$

- ensemble average of Klimontovich f_N, the N-body problem.
- dropping collision terms ~1/N
- moments

Gilbert (ApJ 152, 1968)
Binney, Tremaine (1987)
Bertschinger astro-ph/9503125

$$M^{(n)}_{i_1\ldots i_n}(\boldsymbol{x}) \equiv \int \mathrm{d}^3u \;\; u_{i_1}\cdots u_{i_n} f(\boldsymbol{x}, \boldsymbol{u})$$

- density
- velocity
- velocity dispersion

$$n(\boldsymbol{x}) = M^{(0)} = e^{C^{(0)}}$$
$$u_i(\boldsymbol{x}) = C^{(1)}_i = M^{(1)}_i / n$$
$$C^{(2)}_{ij}(\boldsymbol{x}) = M^{(2)}_{ij}/n - u_i u_j$$

Vlasov (- Poisson) equation (collisionless Boltzmann)

$$\partial_t f(\boldsymbol{x}, \boldsymbol{u}) = -\frac{\boldsymbol{u}}{a^2}\boldsymbol{\nabla}_x f + \boldsymbol{\nabla}_x \Phi \boldsymbol{\nabla}_u f$$

$$\Delta\Phi = \frac{4\pi G\rho_0}{a}\left(\int \mathrm{d}^3u \; f - 1\right)$$

nonlinearity

cosmological scale factor

$$\int_{\mathrm{vol}} \mathrm{d}^3x \int \mathrm{d}^3u \; f = \mathrm{vol}$$

Boltzmann hierarchy

$$\partial_t C^{(n)}_{i_1\cdots i_n} = -\frac{1}{a^2}\left\{\nabla_j C^{(n+1)}_{i_1\cdots i_n j} + \sum_{S\in\mathcal{P}(\mathcal{I}=\{i_1,\cdots,i_n\})} C^{(n+1-|S|)}_{\mathcal{I}\setminus S\cap\{j\}} \nabla_j C^{(|S|)}_S\right\} - \delta_{n1}\,\nabla_{i_1}\Phi$$

Uhlemann, MK, Haugg 1403.5567

consistent truncation:
dust (pressureless perfect) **fluid**

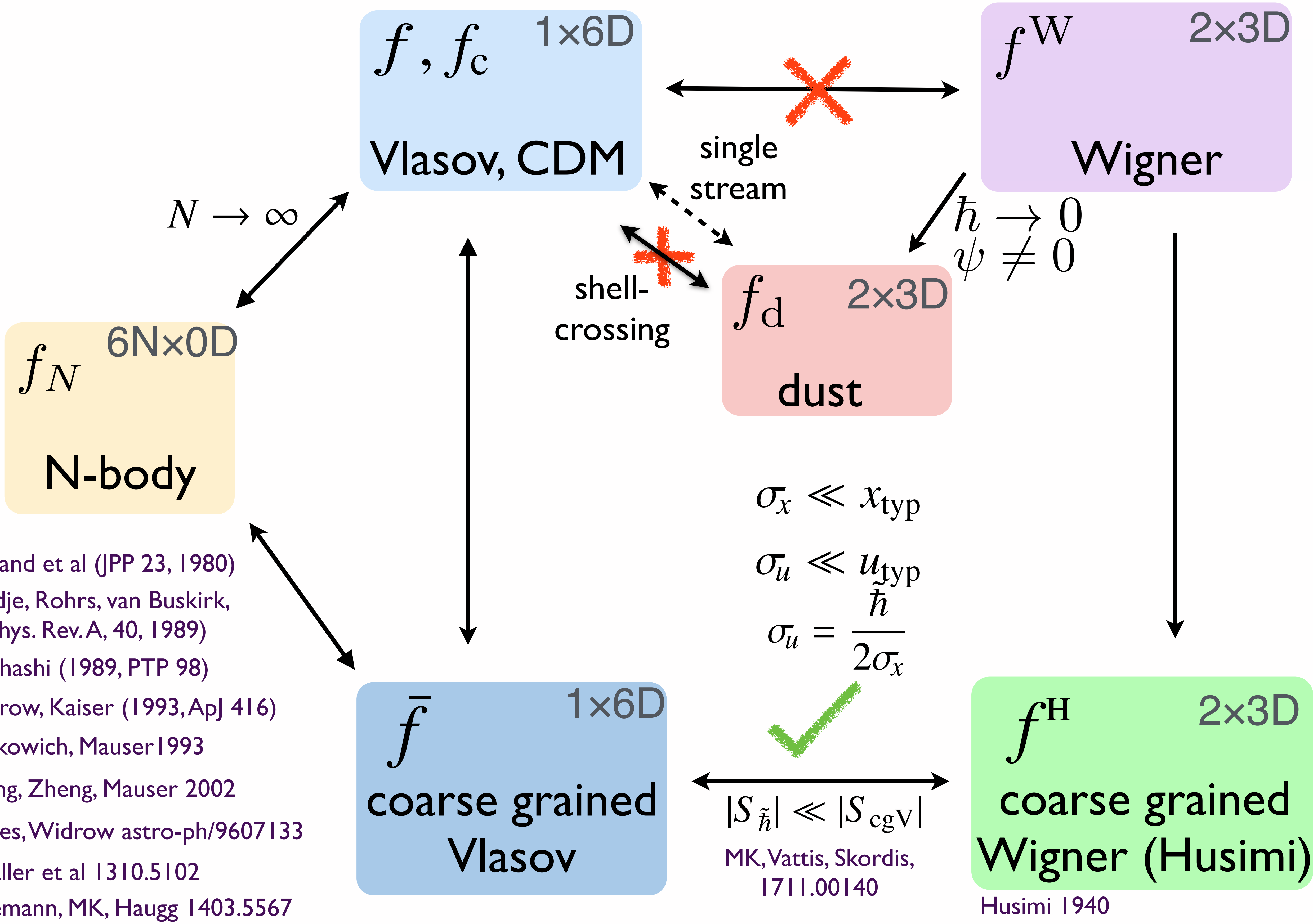
$$C^{(n\geq 2)}(\boldsymbol{x}) = 0$$

$$f_{\mathrm{d}}(t, \boldsymbol{x}, \boldsymbol{u}) = n_{\mathrm{d}}(t, \boldsymbol{x})\,\delta_{\mathrm{D}}\big(\boldsymbol{u} - \boldsymbol{\nabla}\phi_{\mathrm{d}}(t, \boldsymbol{x})\big)$$

Definition of Cold Dark Matter (CDM)

$$\lim_{t\to 0} f_{\mathrm{c}}(t, \boldsymbol{x}, \boldsymbol{u}) = f_{\mathrm{d}}(t, \boldsymbol{x}, \boldsymbol{u})$$

For the purpose of large scale structure formation in cosmology

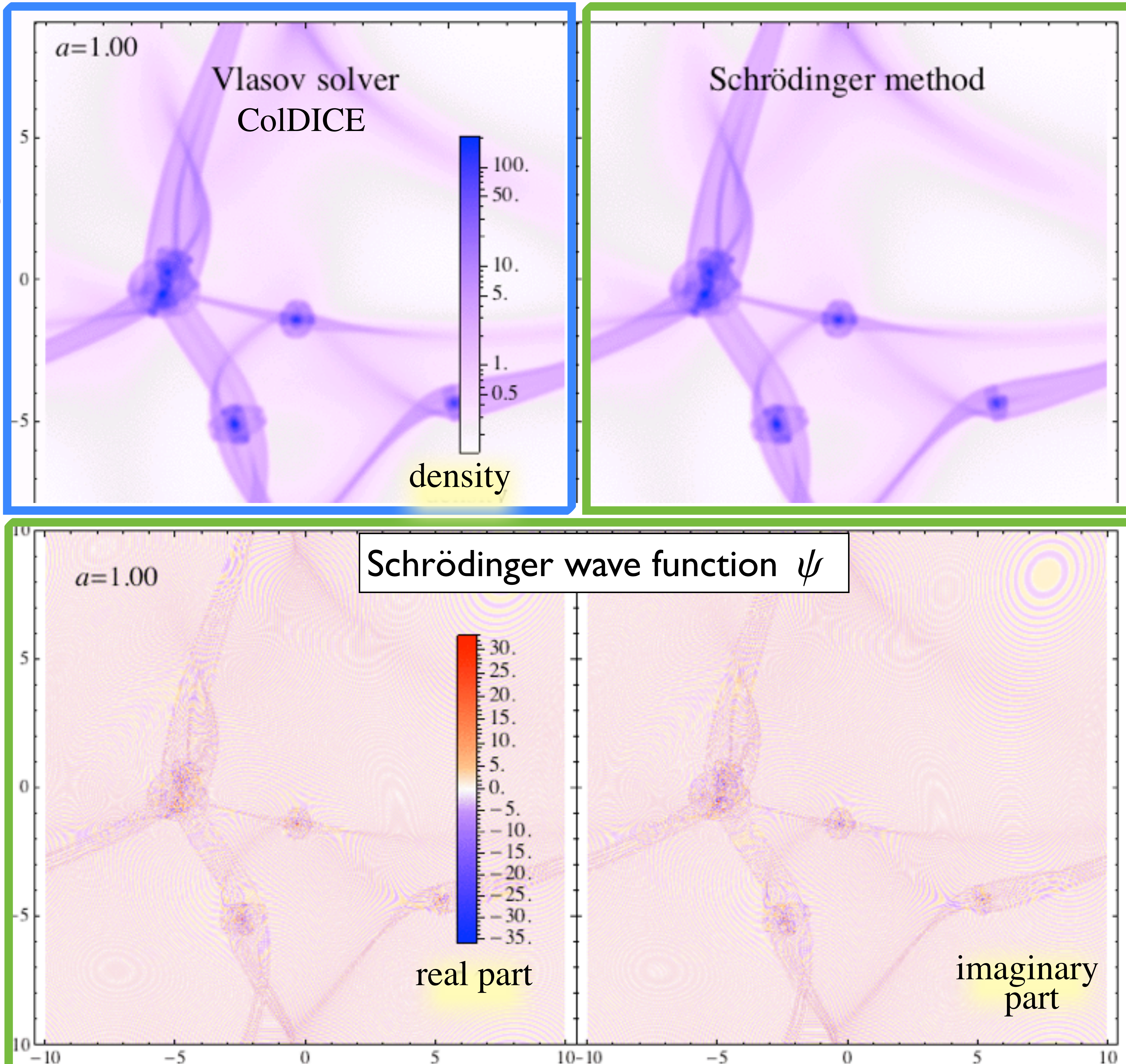


Betrand et al (JPP 23, 1980)
Skodje, Rohrs, van Buskirk, (Phys. Rev. A, 40, 1989)
Takahashi (1989, PTP 98)
Widrow, Kaiser (1993, ApJ 416)
Markowich, Mauser 1993
Zhang, Zheng, Mauser 2002
Davies, Widrow astro-ph/9607133
Schaller et al 1310.5102
Uhlemann, MK, Haugg 1403.5567
Garny, Konstandin 1710.04846

2a) Comparison of **2D** cosmological **simulations** by eye

for coarse grained `ColDICE` and ScM

Sousbie, Colombi, (JCPH,321, 644, 2016) 1509.07720

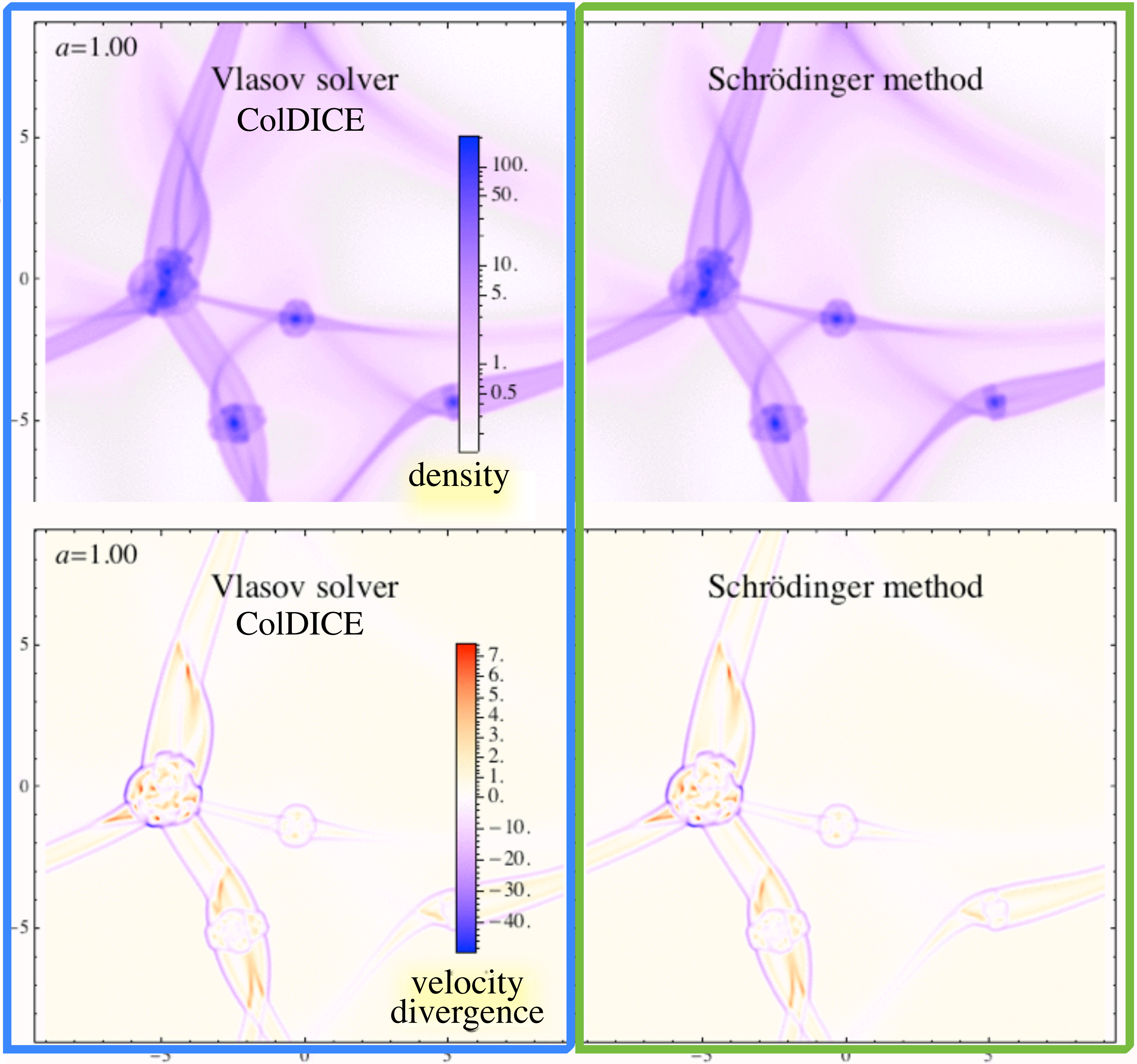


MK, Vattis, Skordis, 1711.00140

$|\psi|^2$
+gaussian filter with width σ_x

- CUDA
- Crank-Nicolson
- CuFFT
- $N=8192^2$

Sousbie, Colombi, (JCPH,321, 644, 2016) 1509.07720



MK, Vattis, Skordis, 1711.00140

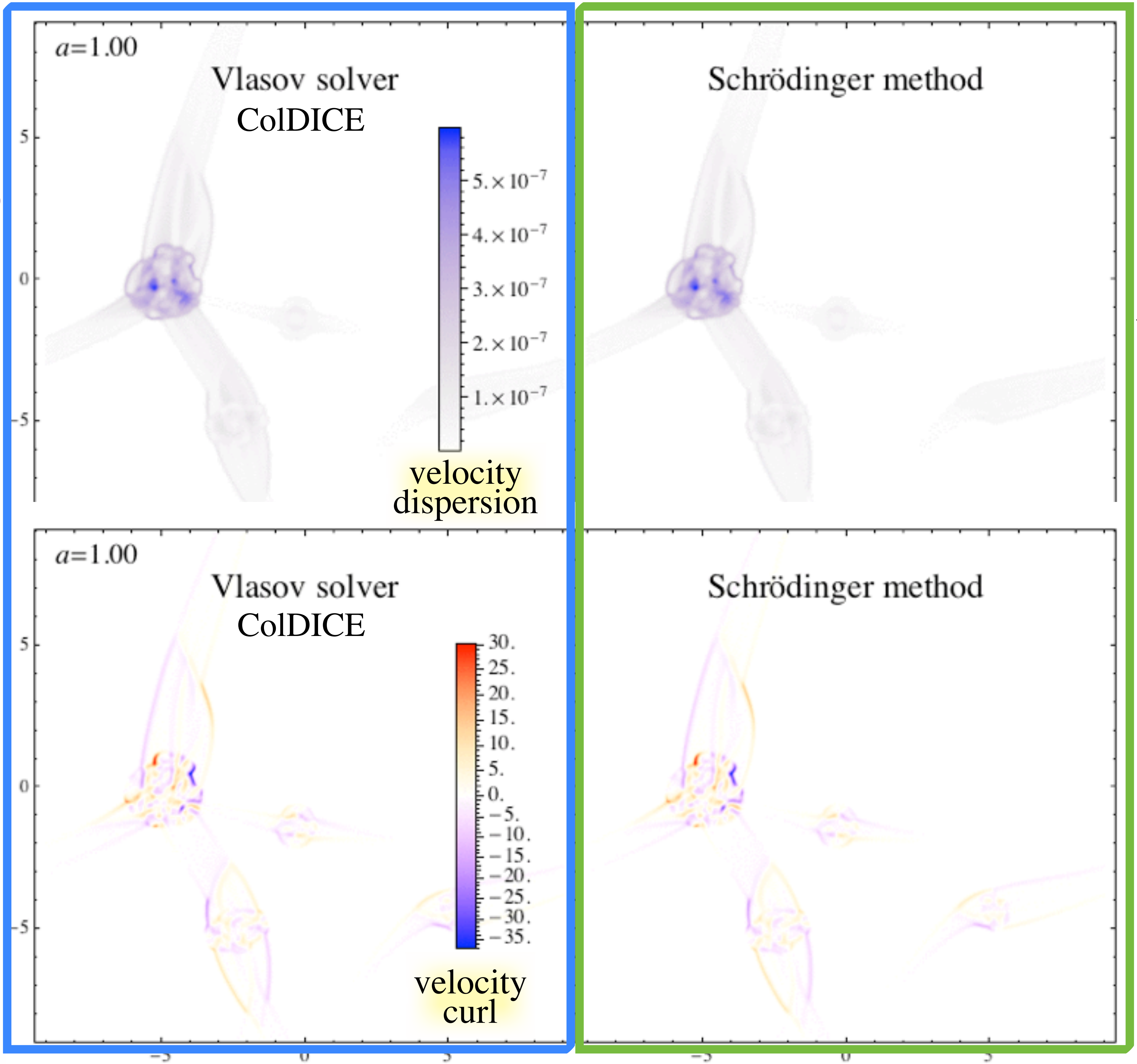
$$M^{\mathrm{w}(0)} = |\psi|^2$$

+gaussian filter with width σ_x

$$M_i^{\mathrm{w}(1)} = \tilde{\hbar}\, \Im\left\{\psi_{,i}\bar{\psi}\right\}$$

Sousbie, Colombi, (JCPH,321, 644, 2016) 1509.07720

MK, Vattis, Skordis, 1711.00140



$$M^{\mathrm{w}(2)}_{ij} = \frac{\tilde{\hbar}^2}{2}\Re\left\{\psi_{,i}\bar{\psi}_{,j} - \psi_{,ij}\bar{\psi}\right\}$$

+gaussian filter with width σ_x, σ_u

$$M^{\mathrm{w}(1)}_{i} = \tilde{\hbar}\,\Im\left\{\psi_{,i}\bar{\psi}\right\}$$

2b) Comparison
of
mathematical formulations
and **1D example**

of coarse grained CDM and ScM

coarse grained CDM and Husimi phase space density

Degrees of freedom: 2×d

$$\mathbb{R}^d \to \mathbb{R}^{2\times d} : \quad \boldsymbol{X}(\boldsymbol{q}),\, \boldsymbol{U}(\boldsymbol{q})$$

Dynamics: 2 non-localities

$$a^2 \partial_t \boldsymbol{X}(\boldsymbol{q}) = \boldsymbol{U}(\boldsymbol{q})$$ Hamiltonian equations

$$\partial_t \boldsymbol{U}(\boldsymbol{q}) = -\boldsymbol{\nabla}_x \Phi_{\mathrm{c}}(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{X}(\boldsymbol{q})}$$

$$\Delta \Phi_{\mathrm{c}}(\boldsymbol{x}) = \frac{4\pi G \rho_0}{a}\Bigg(\sum_{\substack{\boldsymbol{q} \text{ with} \\ \boldsymbol{x}=\boldsymbol{X}(\boldsymbol{q})}} \frac{1}{|\det \partial_{q^i} X^j(\boldsymbol{q})|} - 1\Bigg)$$

Poisson equation — sum over streams

Phase space distr.: non-local

$$f_{\mathrm{c}}(\boldsymbol{x},\boldsymbol{u}) = \int \mathrm{d}^d q \; \delta_{\mathrm{D}}[\boldsymbol{x} - \boldsymbol{X}(\boldsymbol{q})]\, \delta_{\mathrm{D}}[\boldsymbol{u} - \boldsymbol{U}(\boldsymbol{q})]$$

Gaussian smoothing with width σ_x and $\sigma_u = \tilde{\hbar}/(2\sigma_x)$

$$\bar{f}_{\mathrm{c}}(\boldsymbol{x},\boldsymbol{u}) = \int \mathrm{d}^d q \; \frac{e^{-\frac{[\boldsymbol{x}-\boldsymbol{X}(\boldsymbol{q})]^2}{2\sigma_x{}^2}}}{(2\pi)^{d/2}\sigma_x{}^d}\; \frac{e^{-\frac{[\boldsymbol{u}-\boldsymbol{U}(\boldsymbol{q})]^2}{2\sigma_u{}^2}}}{(2\pi)^{d/2}\sigma_u{}^d}$$

2

$$\mathbb{R}^d \to \mathbb{R}^2 : \quad \Re\{\psi(\boldsymbol{x})\},\, \Im\{\psi(\boldsymbol{x})\}$$

1 non-locality

Hamiltonian equations — Arriola, Soler, (JSP, 103, 2001),

$$i\tilde{\hbar}\partial_t \psi(\boldsymbol{x}) = -\frac{\tilde{\hbar}^2}{2a^2}\Delta\psi(\boldsymbol{x}) + \Phi_\psi(\boldsymbol{x})\psi(\boldsymbol{x})$$

$$\Delta \Phi_\psi(\boldsymbol{x}) = \frac{4\pi G \rho_0}{a}\left(|\psi(\boldsymbol{x})|^2 - 1\right)$$

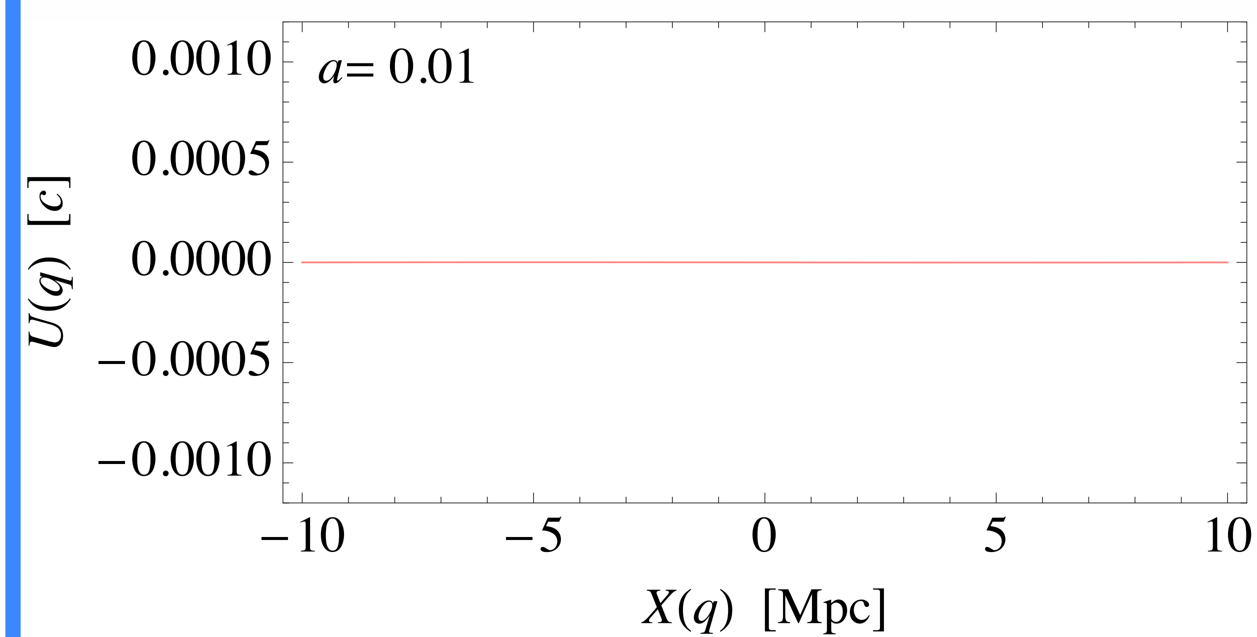
Poisson equation — new parameter: $\tilde{\hbar} = \hbar/m$

quasi-local

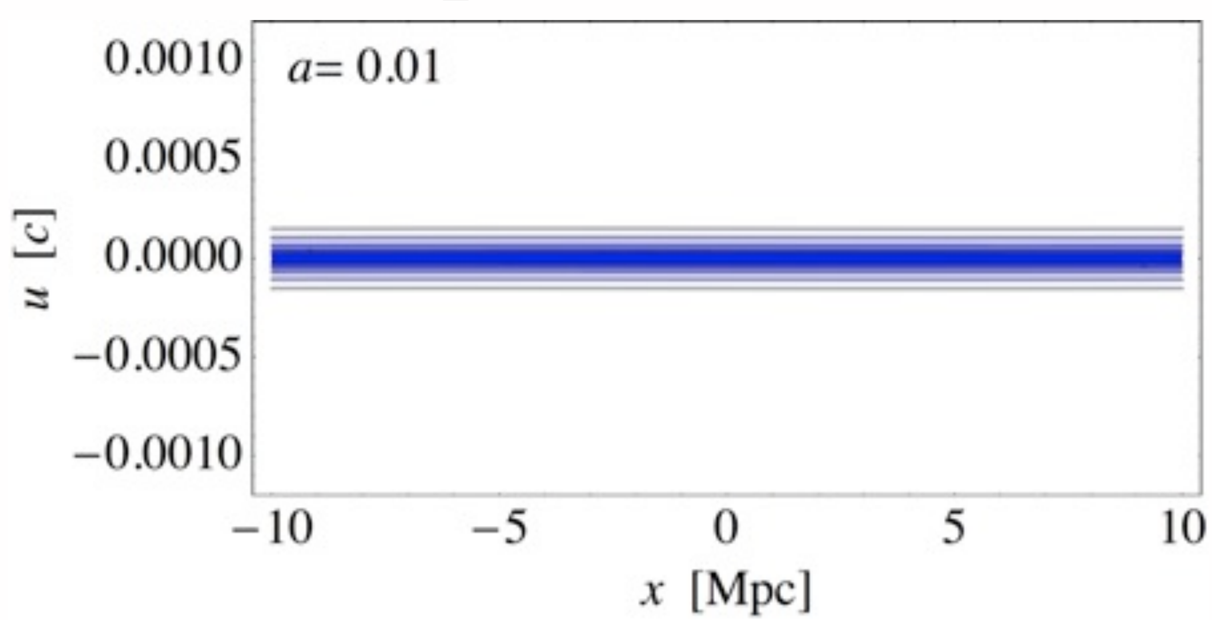
In eulerian space Gaussian filter
has effective range of few σ_x

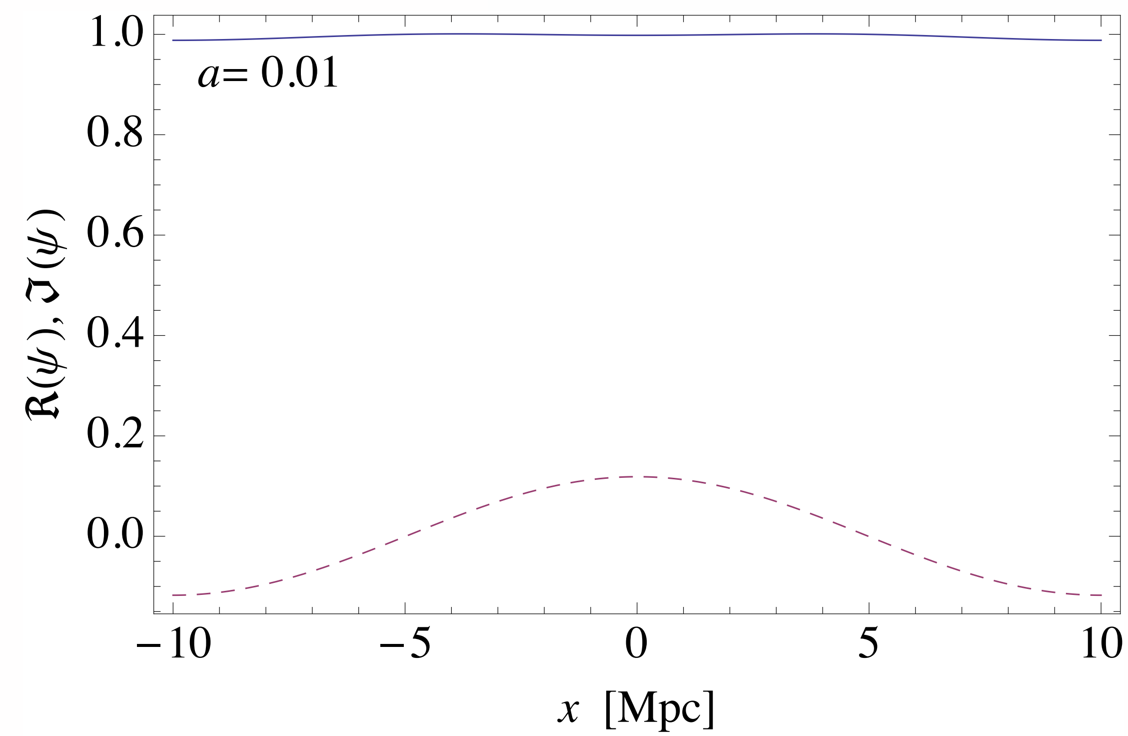
$$f_{\mathrm{H}}(\boldsymbol{x},\boldsymbol{u}) = \left| \int \mathrm{d}^d x' \; \frac{e^{-\frac{(\boldsymbol{x}-\boldsymbol{x}')^2}{4\sigma_x{}^2} - \frac{i}{\tilde{\hbar}}\boldsymbol{u}\cdot\boldsymbol{x}'}}{(2\pi\tilde{\hbar})^{d/2}\,(2\pi\sigma_x{}^2)^{d/4}}\, \psi(\boldsymbol{x}')\right|^2$$

1D pancake collapse: coarse grained CDM and ScM

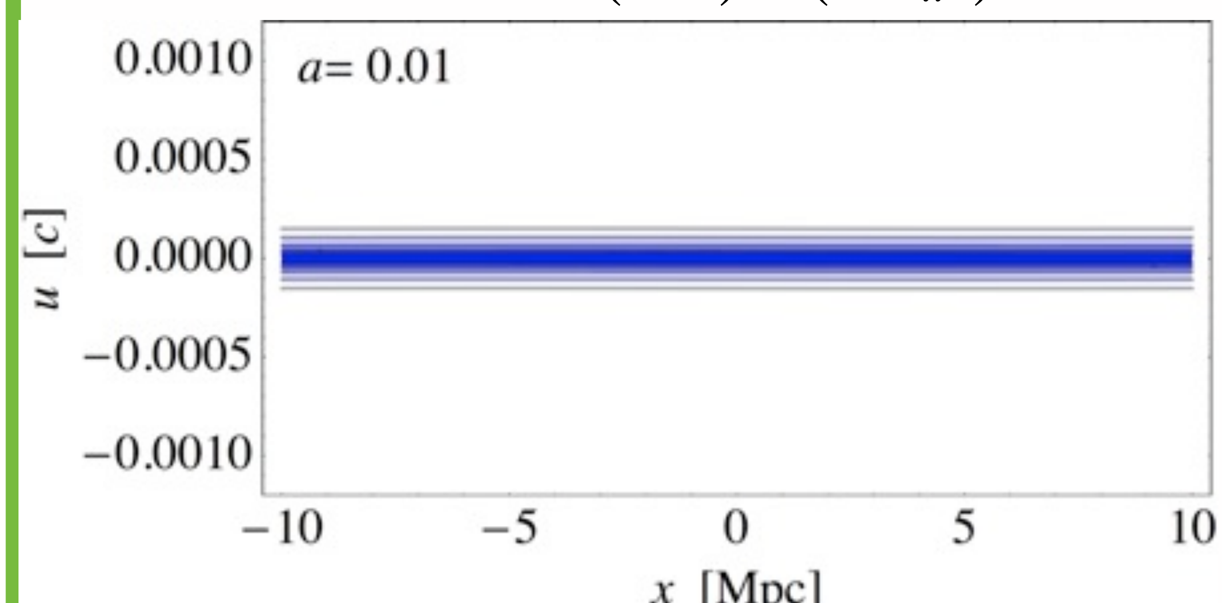


$$\bar{f}_{\mathrm{c}}(x,u) = \int_{-L}^{L} \frac{\mathrm{d}q}{2\pi\sigma_x\sigma_u}\, e^{-\frac{(u-U(q))^2}{2\sigma_u{}^2}}\, e^{-\frac{(x-X(q))^2}{2\sigma_x{}^2}}$$

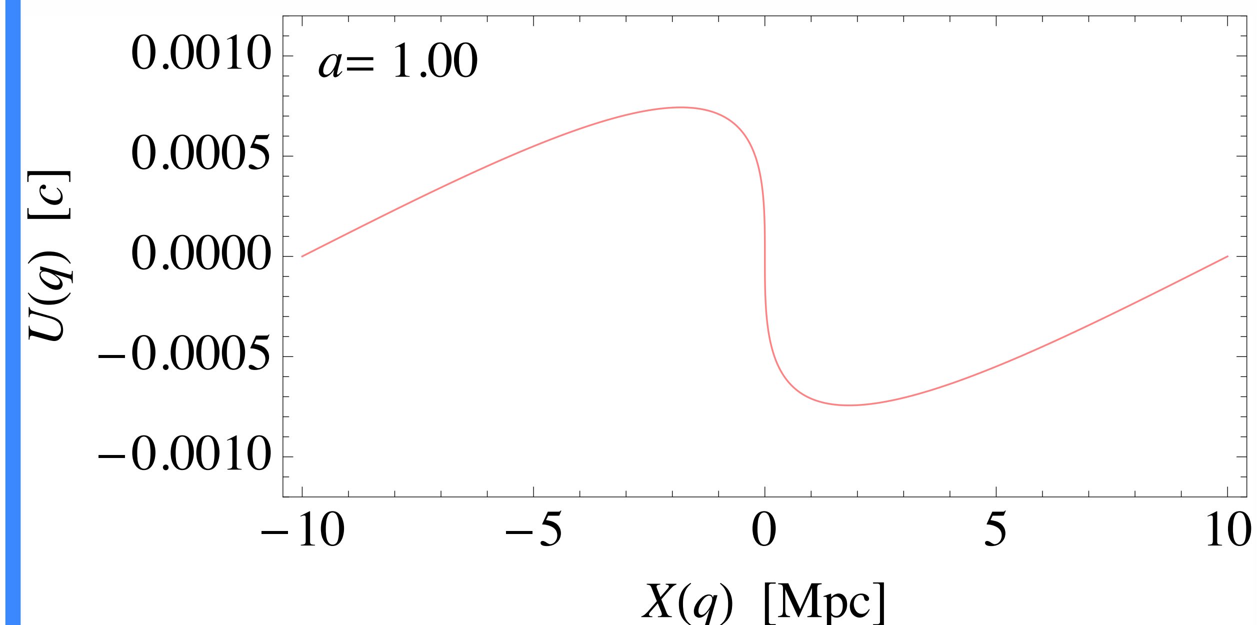




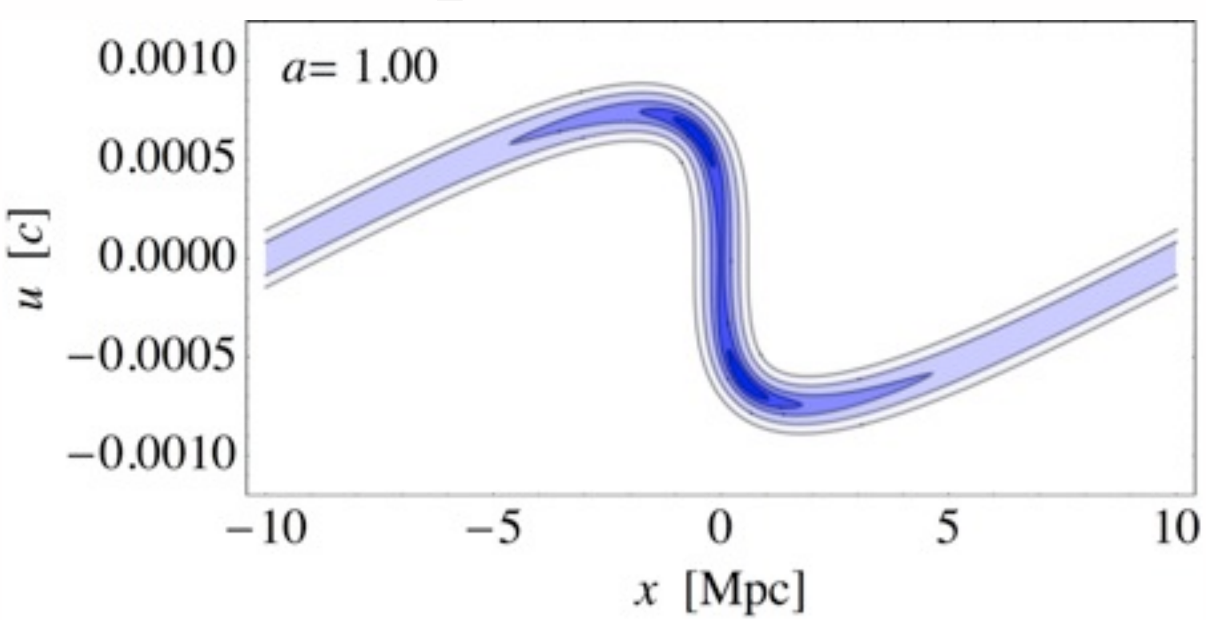
$$f_{\mathrm{H}}(x,u) = \left| \int_{x-5\sigma_x}^{x+5\sigma_x} \mathrm{d}x' \; \frac{e^{-\frac{(x-x')^2}{4\sigma_x{}^2} - \frac{i}{\tilde{\hbar}} u x'}}{(2\pi\tilde{\hbar})^{1/2}(2\pi\sigma_x{}^2)^{1/4}}\, \psi(x') \right|^2$$

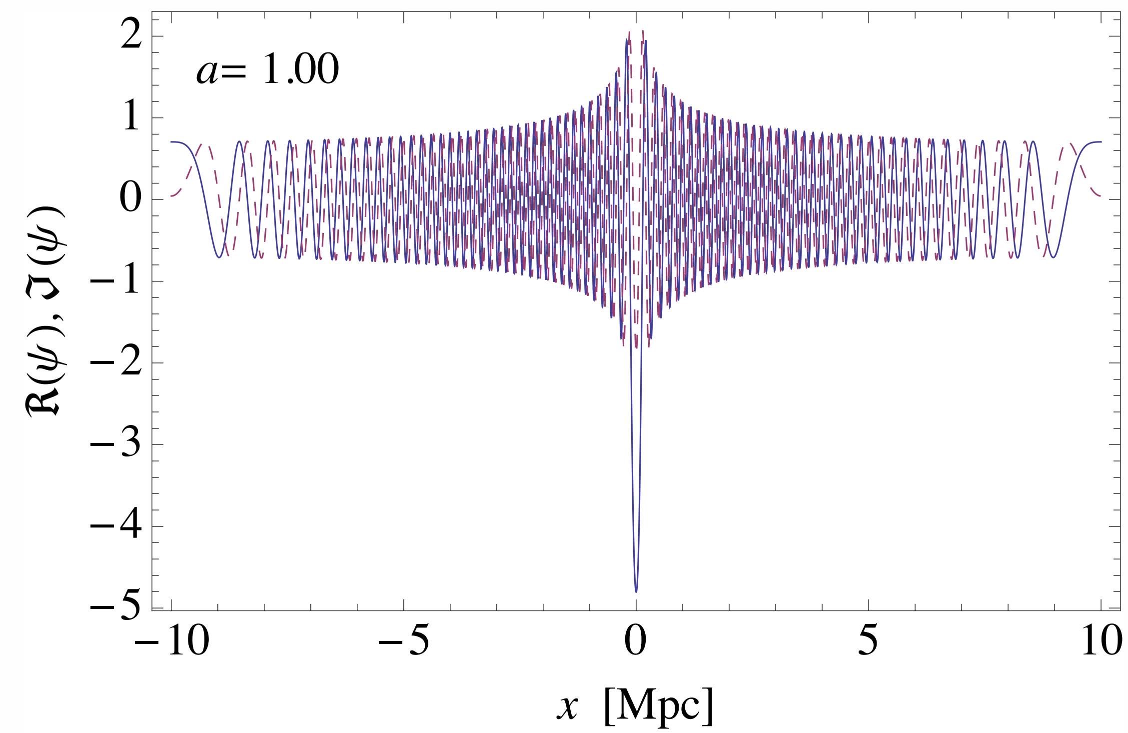


1D pancake collapse: coarse grained CDM and ScM

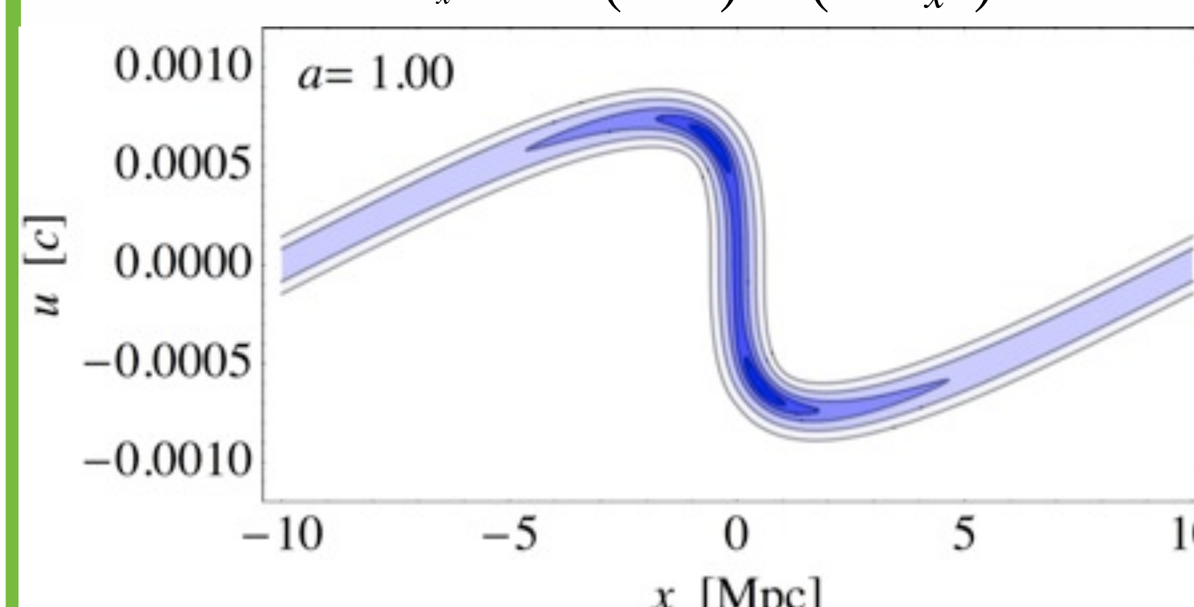


$$\bar{f}_{c}(x,u) = \int_{-L}^{L} \frac{dq}{2\pi\sigma_x\sigma_u}\, e^{-\frac{(u-U(q))^2}{2\sigma_u{}^2}}\, e^{-\frac{(x-X(q))^2}{2\sigma_x{}^2}}$$

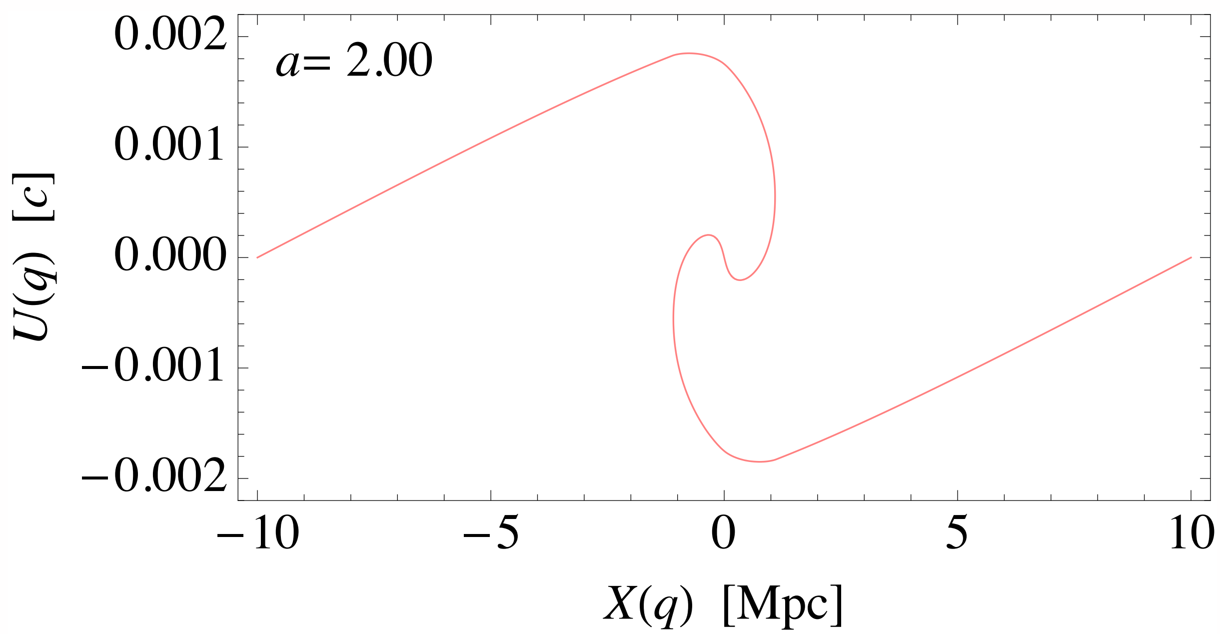




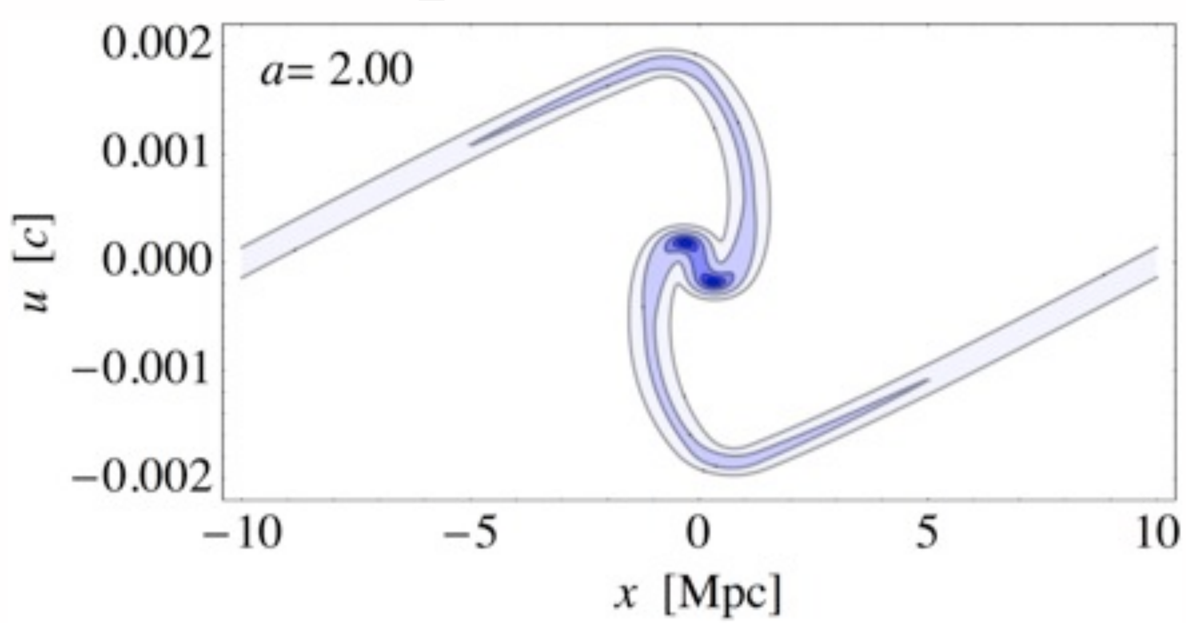
$$f_{H}(x,u) = \left| \int_{x-5\sigma_x}^{x+5\sigma_x} dx' \; \frac{e^{-\frac{(x-x')^2}{4\sigma_x{}^2} - \frac{i}{\tilde{\hbar}} u x'}}{(2\pi\tilde{\hbar})^{1/2}(2\pi\sigma_x{}^2)^{1/4}}\, \psi(x') \right|^2$$

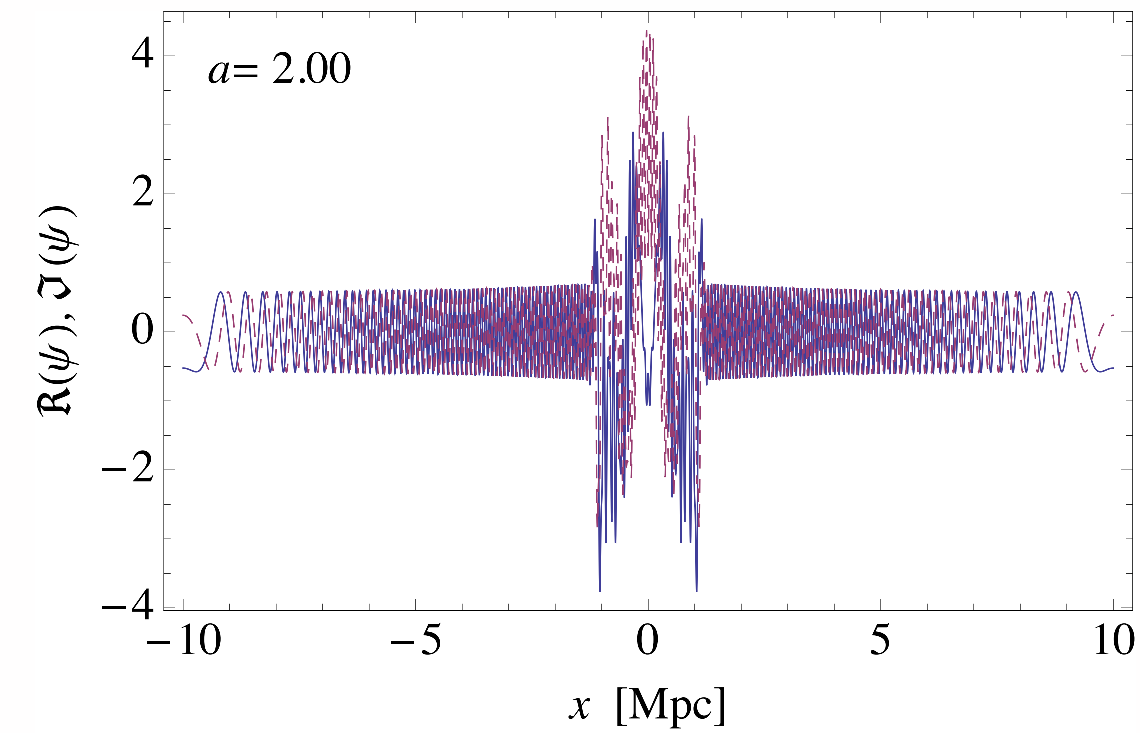


1D pancake collapse: coarse grained CDM and ScM

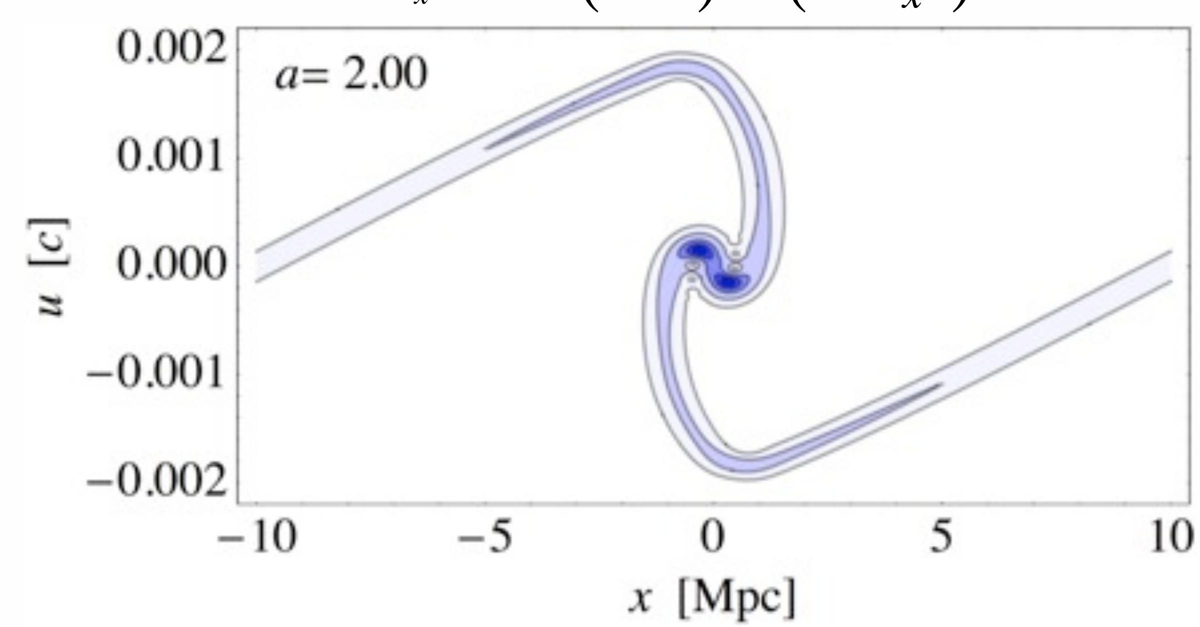


$$\bar{f}_{\mathrm{c}}(x,u) = \int_{-L}^{L} \frac{\mathrm{d}q}{2\pi\sigma_x\sigma_u}\, e^{-\frac{(u-U(q))^2}{2\sigma_u{}^2}}\, e^{-\frac{(x-X(q))^2}{2\sigma_x{}^2}}$$

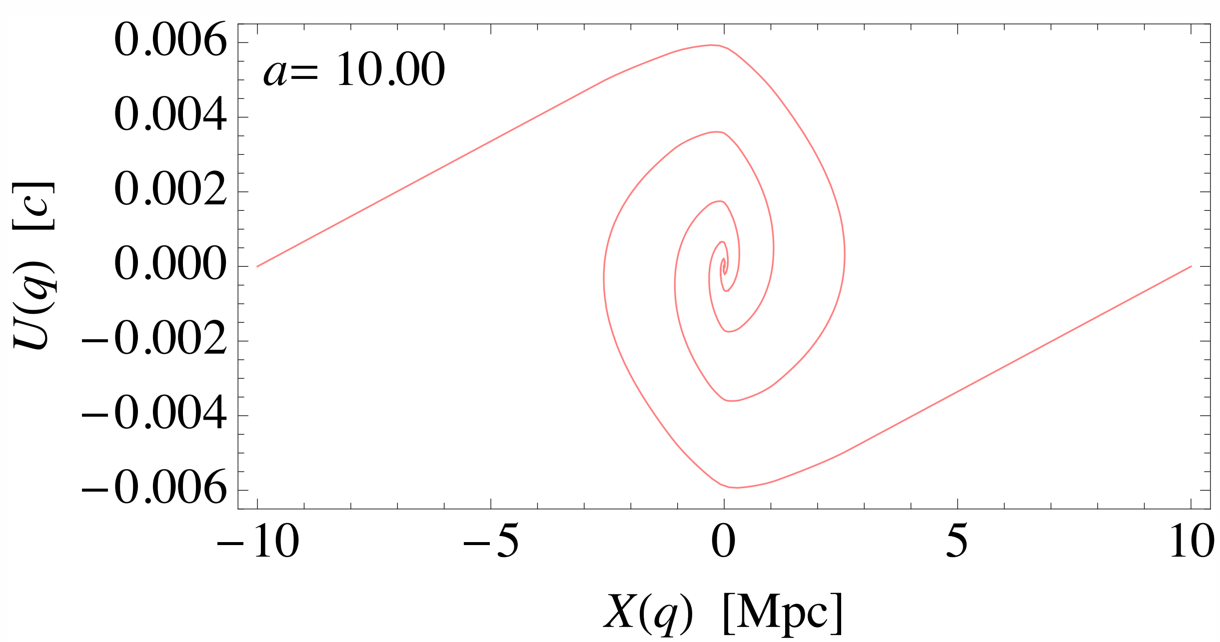




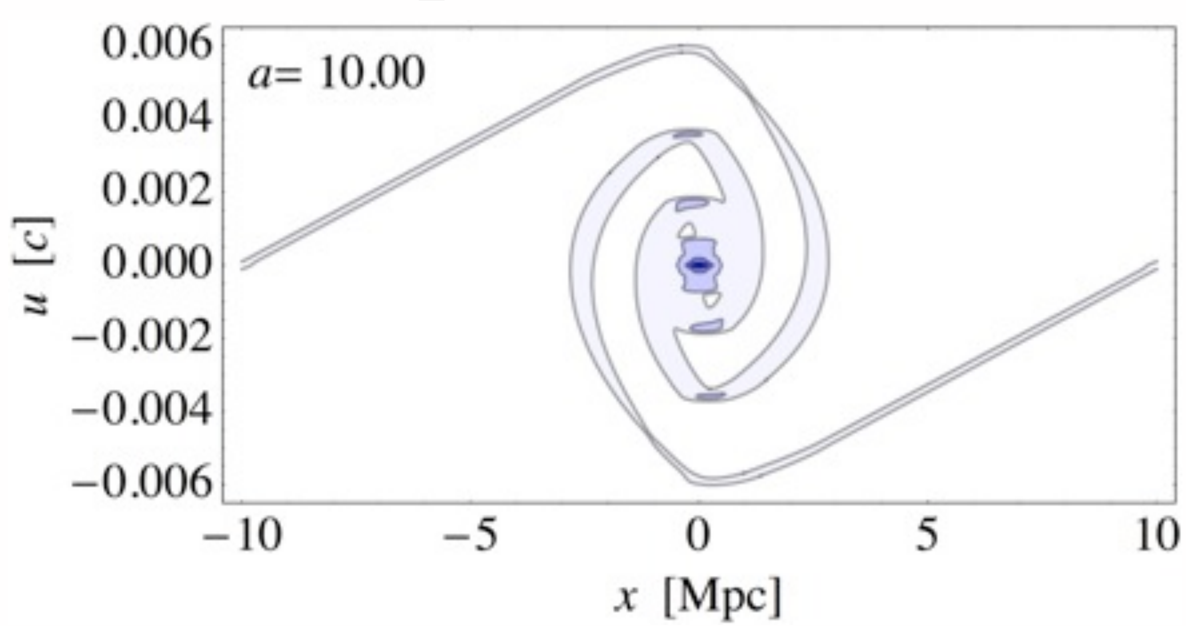
$$f_{\mathrm{H}}(x,u) = \left| \int_{x-5\sigma_x}^{x+5\sigma_x} \mathrm{d}x' \;\; \frac{e^{-\frac{(x-x')^2}{4\sigma_x{}^2} - \frac{i}{\tilde{\hbar}} u x'}}{(2\pi\tilde{\hbar})^{1/2}(2\pi\sigma_x{}^2)^{1/4}}\, \psi(x') \right|^2$$

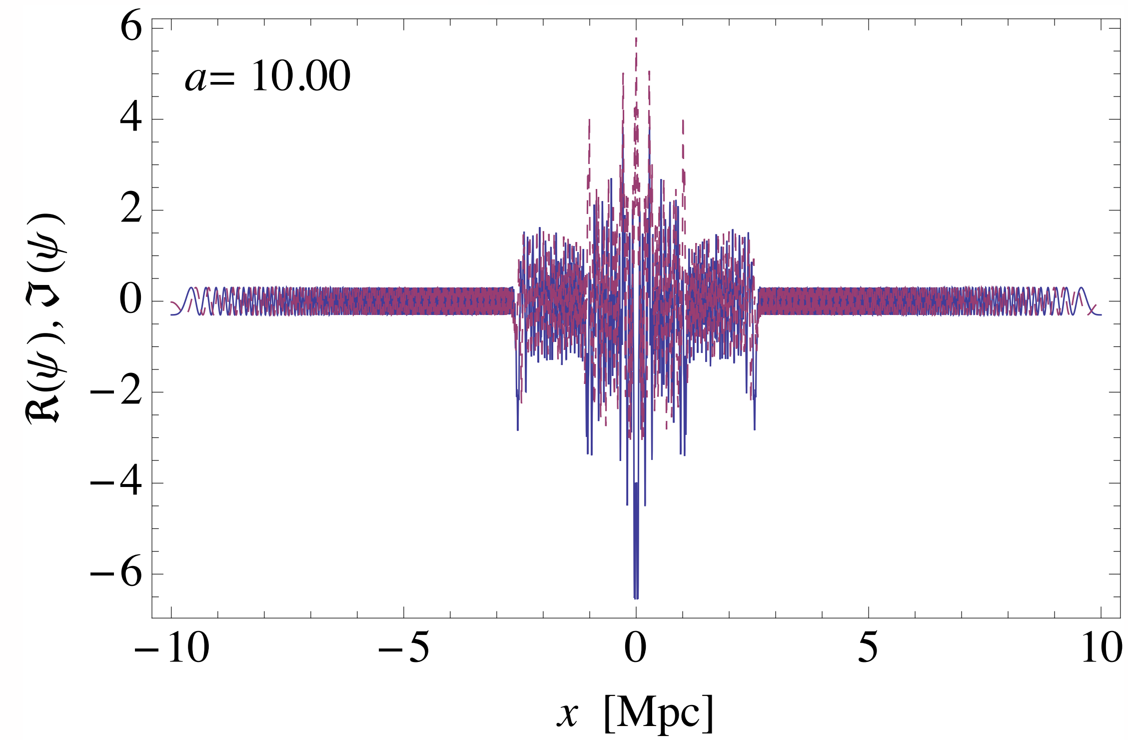


1D pancake collapse: coarse grained CDM and ScM

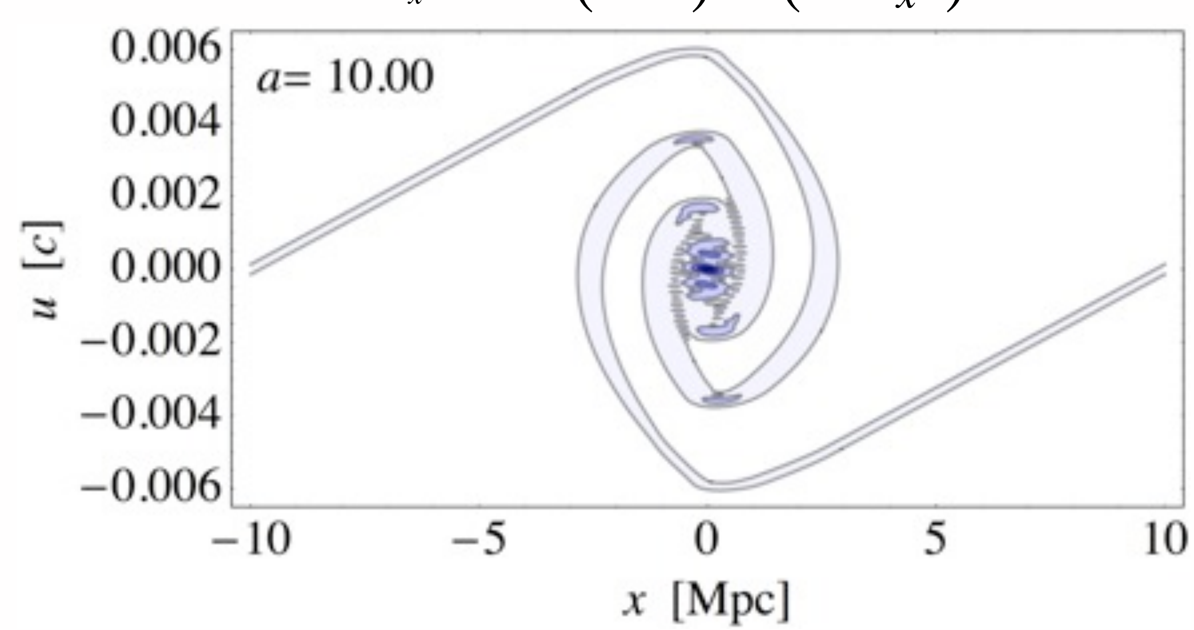


$$\bar{f}_{c}(x,u) = \int_{-L}^{L} \frac{dq}{2\pi\sigma_x\sigma_u}\, e^{-\frac{(u-U(q))^2}{2\sigma_u{}^2}}\, e^{-\frac{(x-X(q))^2}{2\sigma_x{}^2}}$$





$$f_{H}(x,u) = \left| \int_{x-5\sigma_x}^{x+5\sigma_x} dx' \;\; \frac{e^{-\frac{(x-x')^2}{4\sigma_x{}^2} - \frac{i}{\tilde{\hbar}} u x'}}{(2\pi\tilde{\hbar})^{1/2}(2\pi\sigma_x{}^2)^{1/4}}\, \psi(x') \right|^2$$



Convergence of ScM to dust

$$f_{\rm d}(t,\boldsymbol{x},\boldsymbol{u}) = n_{\rm d}(t,\boldsymbol{x})\,\delta_{\rm D}\big(\boldsymbol{u} - \boldsymbol{\nabla}\phi_{\rm d}(t,\boldsymbol{x})\big)$$

↓ Vlasov equation

$$\partial_t n_{\rm d} = -\frac{1}{a^2}\boldsymbol{\nabla}\cdot(n_{\rm d}\boldsymbol{u}_{\rm d})\,,$$

$$\partial_t \boldsymbol{u}_{\rm d} = -\frac{1}{a^2}(\boldsymbol{u}_{\rm d}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{\rm d} - \boldsymbol{\nabla}\Phi_{\rm d}\,,$$

$$\boldsymbol{\nabla}\times\boldsymbol{u}_{\rm d} = 0$$

$$\Delta\Phi_{\rm d} = \frac{4\pi G\rho_0}{a}\big(n_{\rm d} - 1\big)$$

$$\psi(\boldsymbol{x}) =: \sqrt{n_\psi(\boldsymbol{x})}\exp\big(i\phi(\boldsymbol{x})/\tilde{\hbar}\big) \qquad \boldsymbol{u}_\psi \equiv \boldsymbol{\nabla}\phi$$

↓ Schrödinger-Poisson and $n_\psi \neq 0$

$$\partial_t n_\psi = -\frac{1}{a^2}\boldsymbol{\nabla}_x\cdot(n_\psi\boldsymbol{u})$$

$$\partial_t \boldsymbol{u}_\psi = -\frac{1}{a^2}(\boldsymbol{u}_\psi\cdot\boldsymbol{\nabla})\,\boldsymbol{u}_\psi - \boldsymbol{\nabla}\Phi_\psi + \frac{\tilde{\hbar}^2}{2a^2}\boldsymbol{\nabla}\left(\frac{\Delta\sqrt{n_\psi}}{\sqrt{n_\psi}}\right)$$

$$\boldsymbol{\nabla}\times\boldsymbol{u}_\psi = 0$$

$$\Delta\Phi_\psi = \frac{4\pi G\rho_0}{a}\big(n_\psi - 1\big)$$

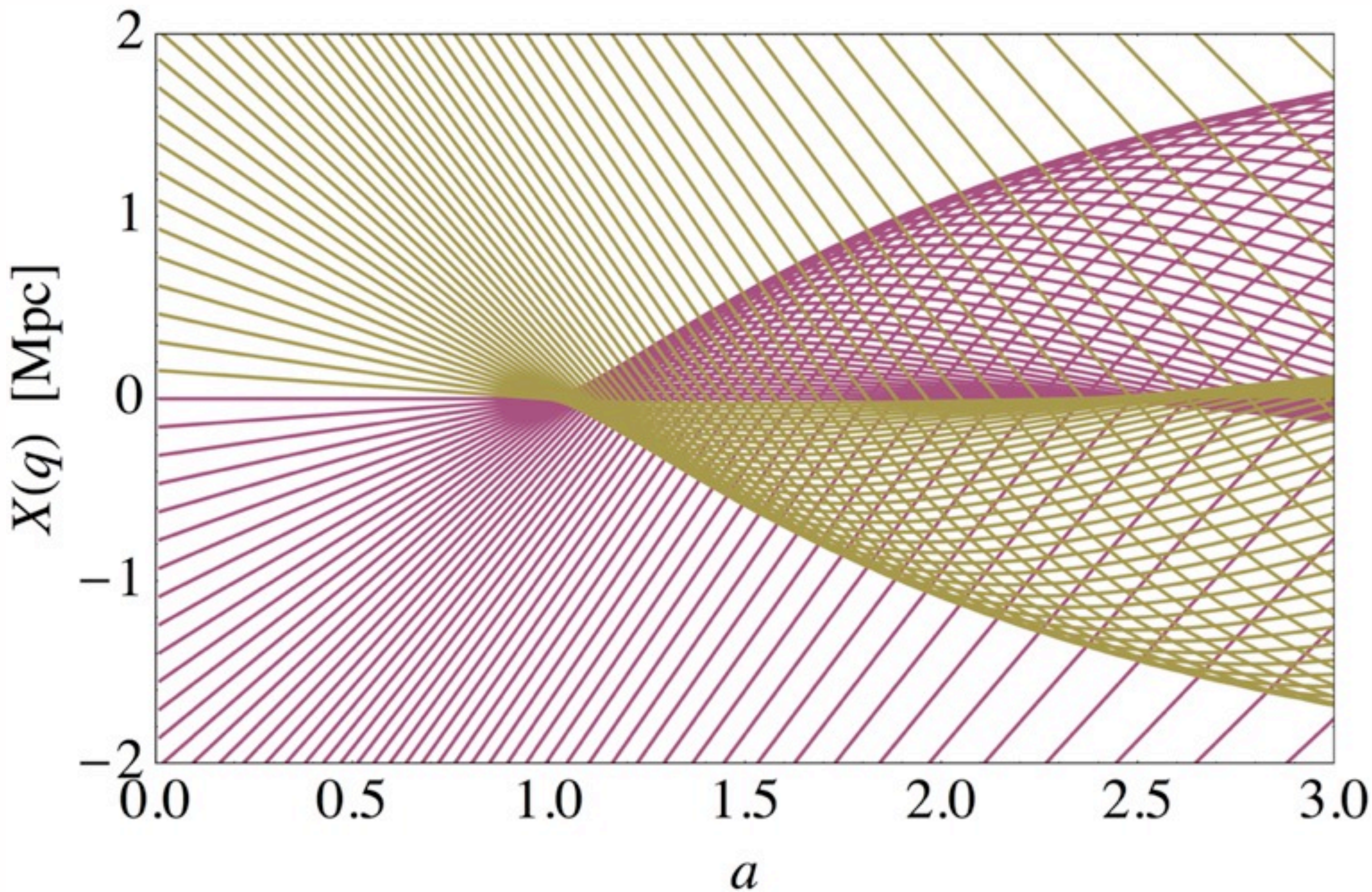
Madelung 1927

$$\left|\frac{\tilde{\hbar}^2}{2a^2}\frac{\Delta\sqrt{n_\psi}}{\sqrt{n_\psi}}\right| \ll \left|\Phi_\psi\right| \quad \Longrightarrow$$ Schrödinger-Poisson becomes dust

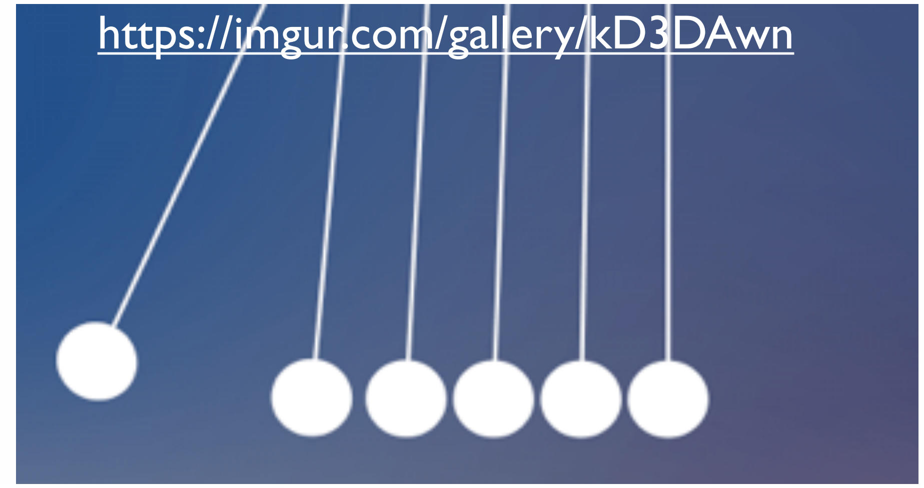
- Useful during single-stream regime to set up initial conditions
- Breakdown of this condition and $n_\psi = 0$ are generic during multi-stream and have no bearing on success of the ScM

shell-crossing : 1D pancake

CDM

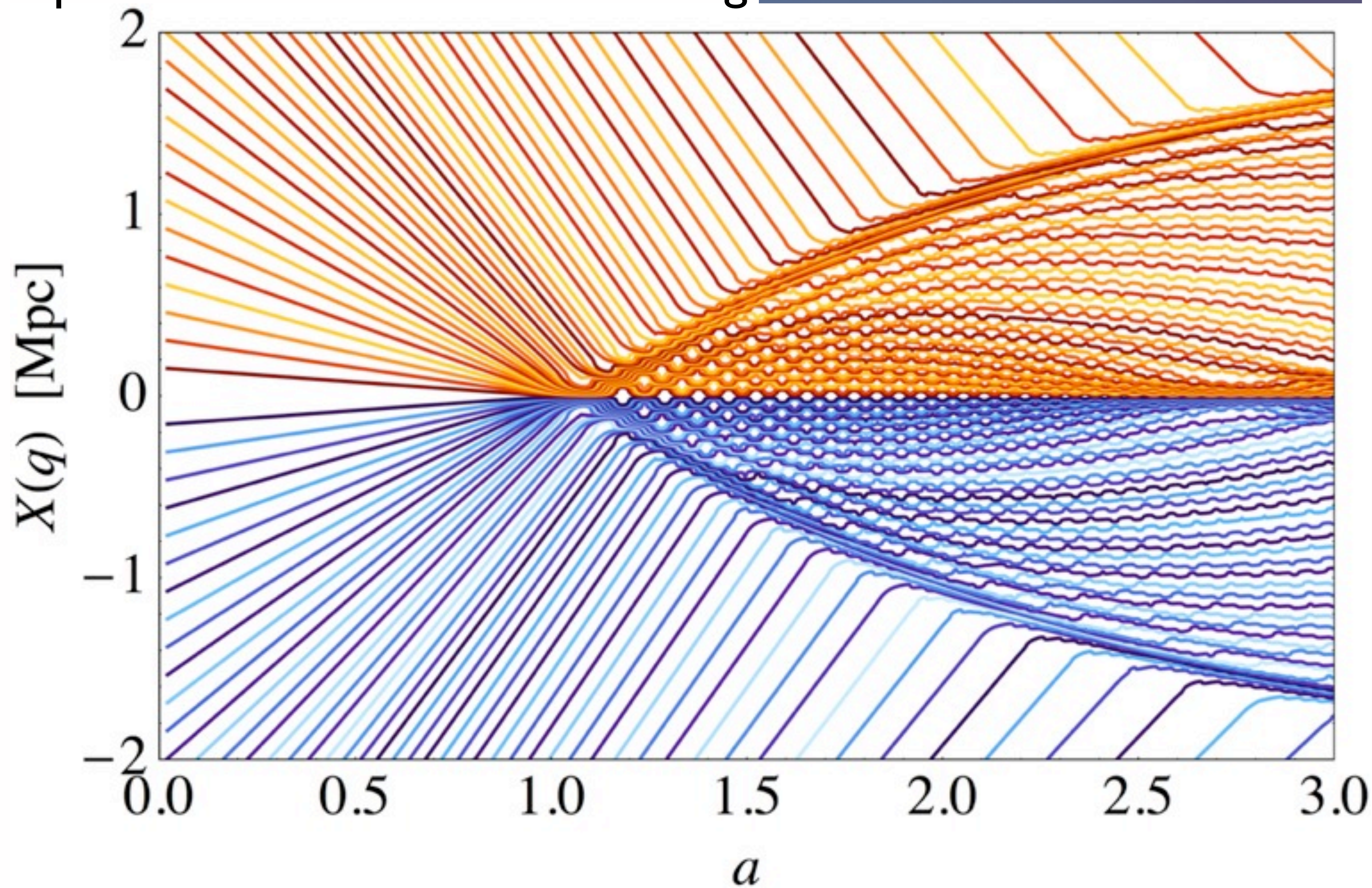


shell-crossing without shell-crossing: 1D pancake



ScM

- Integral lines of $\nabla\phi$: Bohmian trajectories
- Derived concept in ScM (and not needed)
- Quantum pressure emulates shell crossing



Convergence of ScM to coarse grained Vlasov

Coarse grained Vlasov

$$\partial_t \bar{f} = -\frac{\boldsymbol{u}}{a^2}\boldsymbol{\nabla}_x \bar{f} - \frac{{\sigma_u}^2}{a^2}\boldsymbol{\nabla}_x\boldsymbol{\nabla}_u \bar{f} + \boldsymbol{\nabla}_x \bar{\Phi}\exp({\sigma_x}^2\overleftarrow{\boldsymbol{\nabla}}_x\overrightarrow{\boldsymbol{\nabla}}_x)\boldsymbol{\nabla}_u\bar{f}$$

$$\bar{f}(\boldsymbol{x},\boldsymbol{u}) = \int \frac{\mathrm{d}^3x'\,\mathrm{d}^3u'}{(2\pi\sigma_x\sigma_u)^3} e^{-\frac{(\boldsymbol{x}-\boldsymbol{x}')^2}{2{\sigma_x}^2}-\frac{(\boldsymbol{u}-\boldsymbol{u}')^2}{2{\sigma_u}^2}} f(\boldsymbol{x}',\boldsymbol{u}') = e^{\frac{{\sigma_x}^2}{2}\Delta_x+\frac{{\sigma_u}^2}{2}\Delta_u}\{f\} \qquad \sigma_u = \frac{\tilde{\hbar}}{2\sigma_x}$$

Husimi equation (automatically satisfied if Schrödinger-Poisson is solved)

$$\partial_t f_{\mathrm{H}} = -\frac{\boldsymbol{u}}{a^2}\boldsymbol{\nabla}_x f_{\mathrm{H}} - \frac{{\sigma_u}^2}{a^2}\boldsymbol{\nabla}_x\boldsymbol{\nabla}_u f_{\mathrm{H}} + \Phi_{\mathrm{H}}\exp({\sigma_x}^2\overleftarrow{\boldsymbol{\nabla}}_x\overrightarrow{\boldsymbol{\nabla}}_x)\frac{2}{\tilde{\hbar}}\sin\left(\frac{\tilde{\hbar}}{2}\overleftarrow{\boldsymbol{\nabla}}_x\overrightarrow{\boldsymbol{\nabla}}_u\right) f_{\mathrm{H}}$$

$$\partial_t f_{\mathrm{H}} = S_{\mathrm{V}} + S_{\mathrm{cgV}} + S_{\tilde{\hbar}} + O(\tilde{\hbar}^2{\sigma_x}^2, \tilde{\hbar}^4)$$

Necessary to approximate coarse grained Vlasov:

$$|S_{\tilde{\hbar}}| \ll |S_{\mathrm{cgV}}|$$

$$S_{\mathrm{V}} \equiv -\frac{\boldsymbol{u}}{a^2}\cdot\boldsymbol{\nabla}_x f_{\mathrm{H}} + \boldsymbol{\nabla}_x\Phi_{\mathrm{H}}\cdot\boldsymbol{\nabla}_u f_{\mathrm{H}}$$

Vlasov source

$$S_{\mathrm{cgV}} \equiv -\frac{{\sigma_u}^2}{a^2}\boldsymbol{\nabla}_x\cdot\boldsymbol{\nabla}_u f_{\mathrm{H}} + {\sigma_x}^2(\partial_{x_i}\partial_{x_j}\Phi_{\mathrm{H}})(\partial_{x_i}\partial_{u_j}f_{\mathrm{H}}),$$

coarse graining source

$$S_{\tilde{\hbar}} \equiv -\frac{\tilde{\hbar}^2}{24}(\partial_{x_i}\partial_{x_j}\partial_{x_k}\Phi_{\mathrm{H}})(\partial_{u_i}\partial_{u_j}\partial_{u_k}f_{\mathrm{H}})$$

quantum artifact source

coarse grained CDM and Husimi moments

phase space sheet has to be tracked

Dynamics: 2 non-localities

$$a^2\partial_t \boldsymbol{X}(\boldsymbol{q}) = \boldsymbol{U}(\boldsymbol{q})$$

$$\partial_t \boldsymbol{U}(\boldsymbol{q}) = -\boldsymbol{\nabla}_x \Phi_\mathrm{c}(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{X}(\boldsymbol{q})}$$

$$\Delta\Phi_\mathrm{c}(\boldsymbol{x}) = \frac{4\pi G\rho_0}{a}\Big(\sum_{\substack{\boldsymbol{q}\ \text{with}\\ \boldsymbol{x}=\boldsymbol{X}(\boldsymbol{q})}} \frac{1}{|\det \partial_{q^i} X^j(\boldsymbol{q})|} - 1\Big)$$

Moments: non-local

$$G_\mathrm{c}(\boldsymbol{x},\boldsymbol{J}) = \sum_{\substack{\boldsymbol{q}\ \text{with}\\ \boldsymbol{x}=\boldsymbol{X}(t,\boldsymbol{q})}} \frac{e^{i\boldsymbol{J}\cdot\boldsymbol{U}(\boldsymbol{q})}}{|\det \partial_{q^i} X^j(\boldsymbol{q})|}$$

sum over streams

$$\bar{M}^{\mathrm{c}(n)}_{i_1,\ldots,i_n}(\boldsymbol{x}) = e^{\frac{\sigma_x^2}{2}\Delta}\left\{\frac{(-i)^n\partial^n}{\partial J_{i_1}\ldots\partial J_{i_n}} e^{-\frac{1}{2}\sigma_u^2\boldsymbol{J}^2} G_\mathrm{c}(\boldsymbol{x},\boldsymbol{J})\right\}\Bigg|_{\boldsymbol{J}=0}$$

complete avoids phase space

1 non-locality

$$i\tilde{\hbar}\partial_t\psi(\boldsymbol{x}) = -\frac{\tilde{\hbar}^2}{2a^2}\Delta\psi(\boldsymbol{x}) + \Phi_\psi(\boldsymbol{x})\psi(\boldsymbol{x})$$

$$\Delta\Phi_\psi(\boldsymbol{x}) = \frac{4\pi G\rho_0}{a}\big(|\psi(\boldsymbol{x})|^2 - 1\big)$$

quasi-local

$$G_\mathrm{w}(\boldsymbol{x},\boldsymbol{J}) = \psi\left(\boldsymbol{x} + \tfrac{\tilde{\hbar}}{2}\boldsymbol{J}\right)\bar{\psi}\left(\boldsymbol{x} - \tfrac{\tilde{\hbar}}{2}\boldsymbol{J}\right)$$

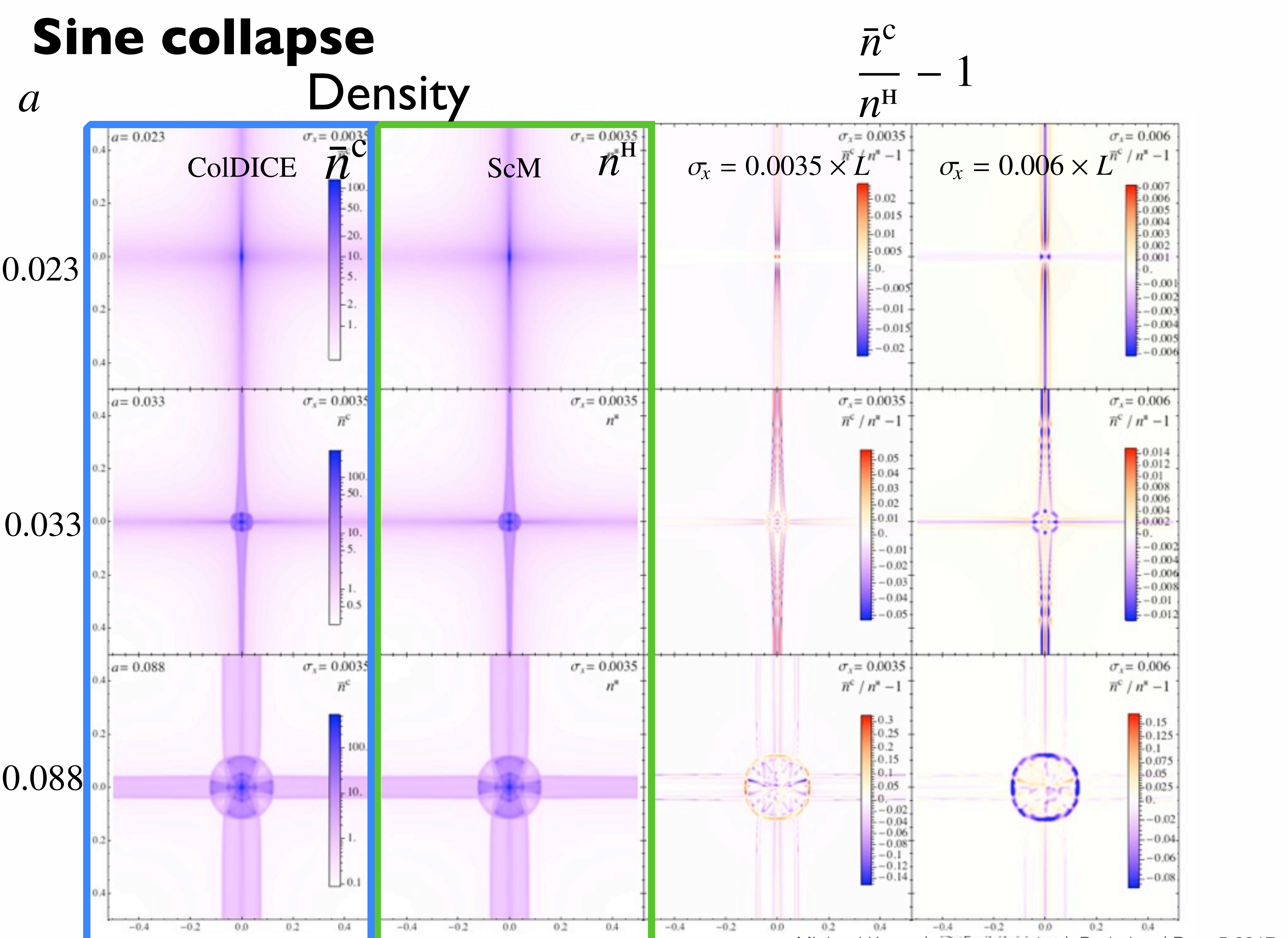
$$M^{\mathrm{H}(n)}_{i_1,\ldots,i_n}(\boldsymbol{x}) = e^{\frac{\sigma_x^2}{2}\Delta}\left\{\frac{(-i)^n\partial^n}{\partial J_{i_1}\ldots\partial J_{i_n}} e^{-\frac{1}{2}\sigma_u^2\boldsymbol{J}^2} G_\mathrm{w}(\boldsymbol{x},\boldsymbol{J})\right\}\Bigg|_{\boldsymbol{J}=0}$$

n derivatives of ψ

2c) Quantitative **Comparison** of **2D** cosmological **simulations**

for ColDICE and ScM

Sine collapse

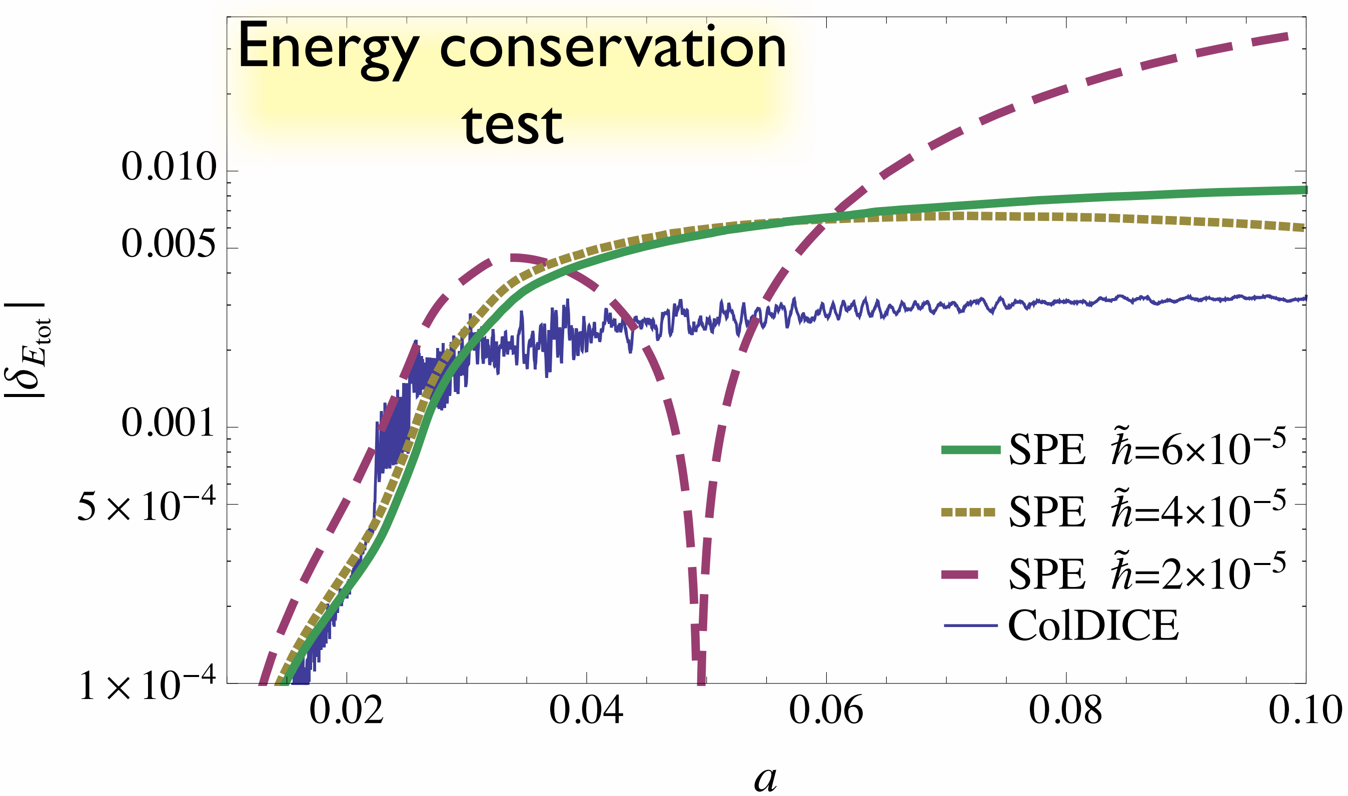


Sine collapse

Numerical convergence

- the larger $\tilde{\hbar}$, the better
- similar to `ColDICE`

$$\delta_{E_{\rm tot}} \equiv \frac{E_{\rm tot}(a)}{E(a_{\rm ini})} - 1 \overset{?}{=} 0$$



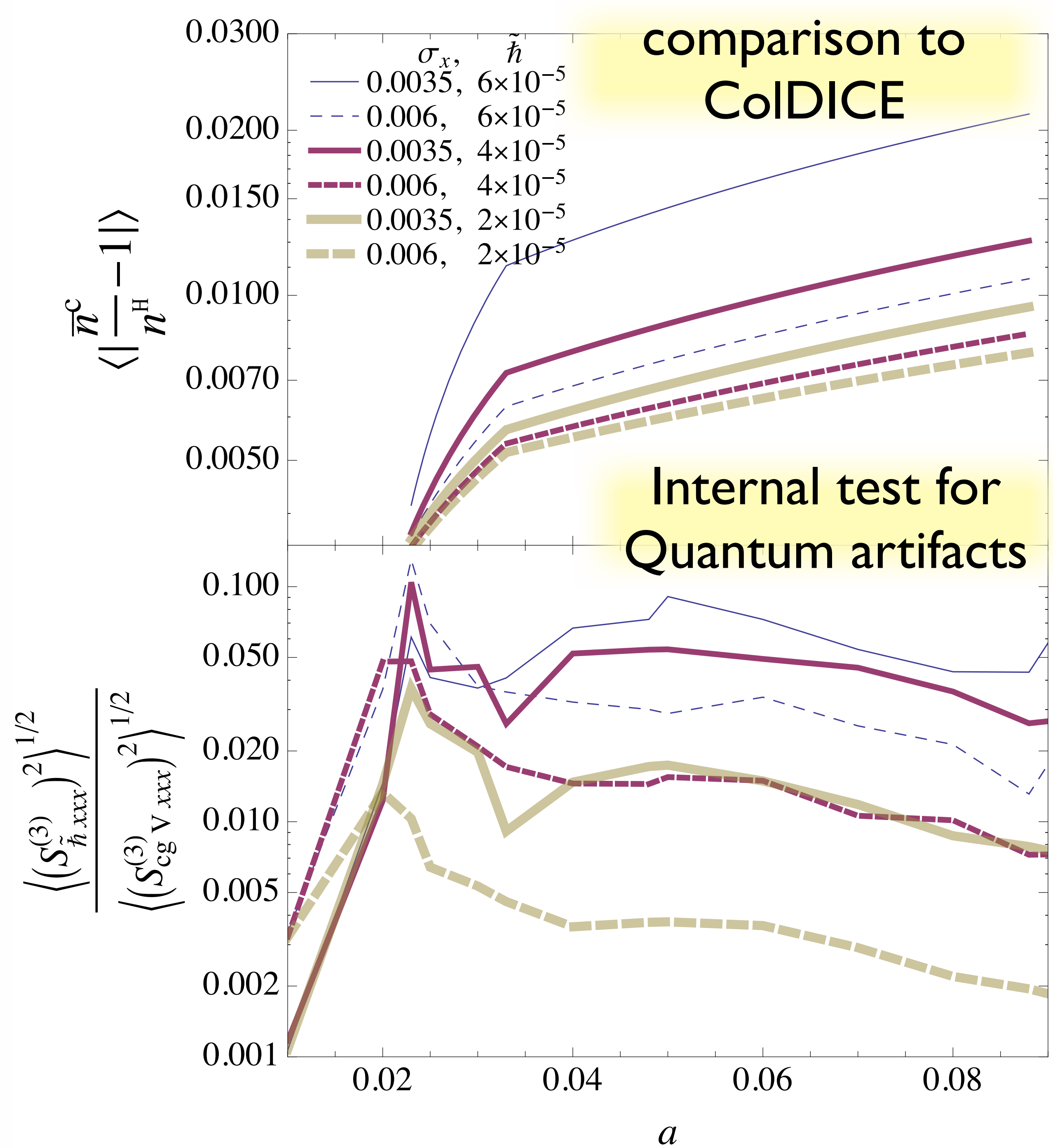
$$K(t) = \frac{\tilde{\hbar}^2}{2a^2}\int d^3x\ |\nabla_x\psi|^2 \qquad E(t) := K(t) + W(t)$$

$$E_{\rm tot} := E(t) + E_{\rm exp}(t)$$

$$W(t) = \frac{1}{2}\int d^3x\ \Phi_\psi\,|\psi|^2 \qquad E_{\rm exp} = \int_{a_{\rm ini}}^{a} \frac{2K(a') + W(a')}{a'}da'$$

Convergence to Vlasov

- the smaller $\tilde{\hbar}$, the better
- $\langle \bar{n}^{\rm C}/n^{\rm H} - 1\rangle \lesssim 1\%$



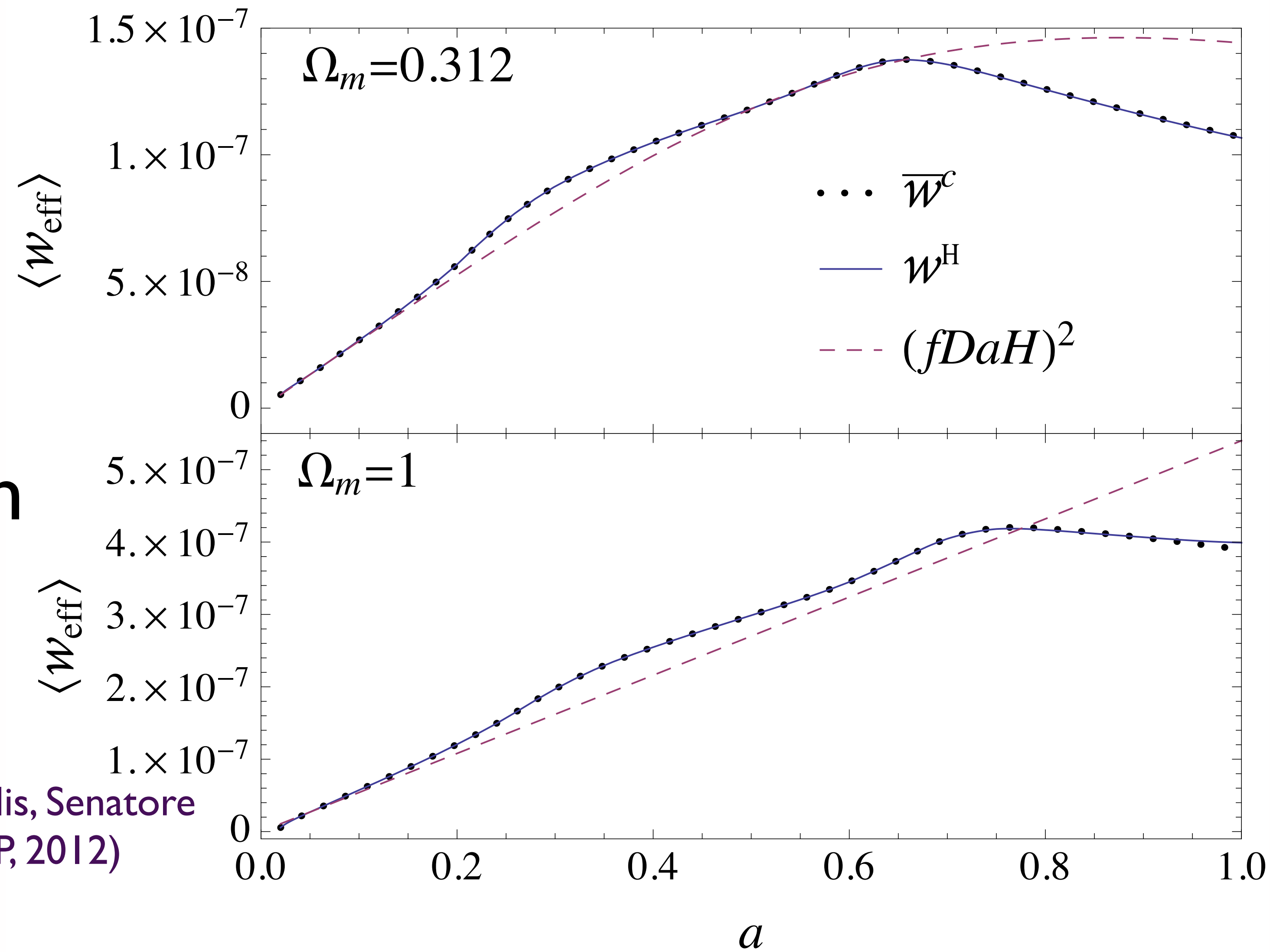
3a) Discussion:
Cosmological backreaction estimate
from CDM and ScM

Effective pressure

$$T^i{}_i = \rho_0 M^{(2)}_{ii} / a^5 = 2P_{\rm eff}$$

$$w_{\rm eff} \equiv P_{\rm eff} a^3 / \rho_0$$

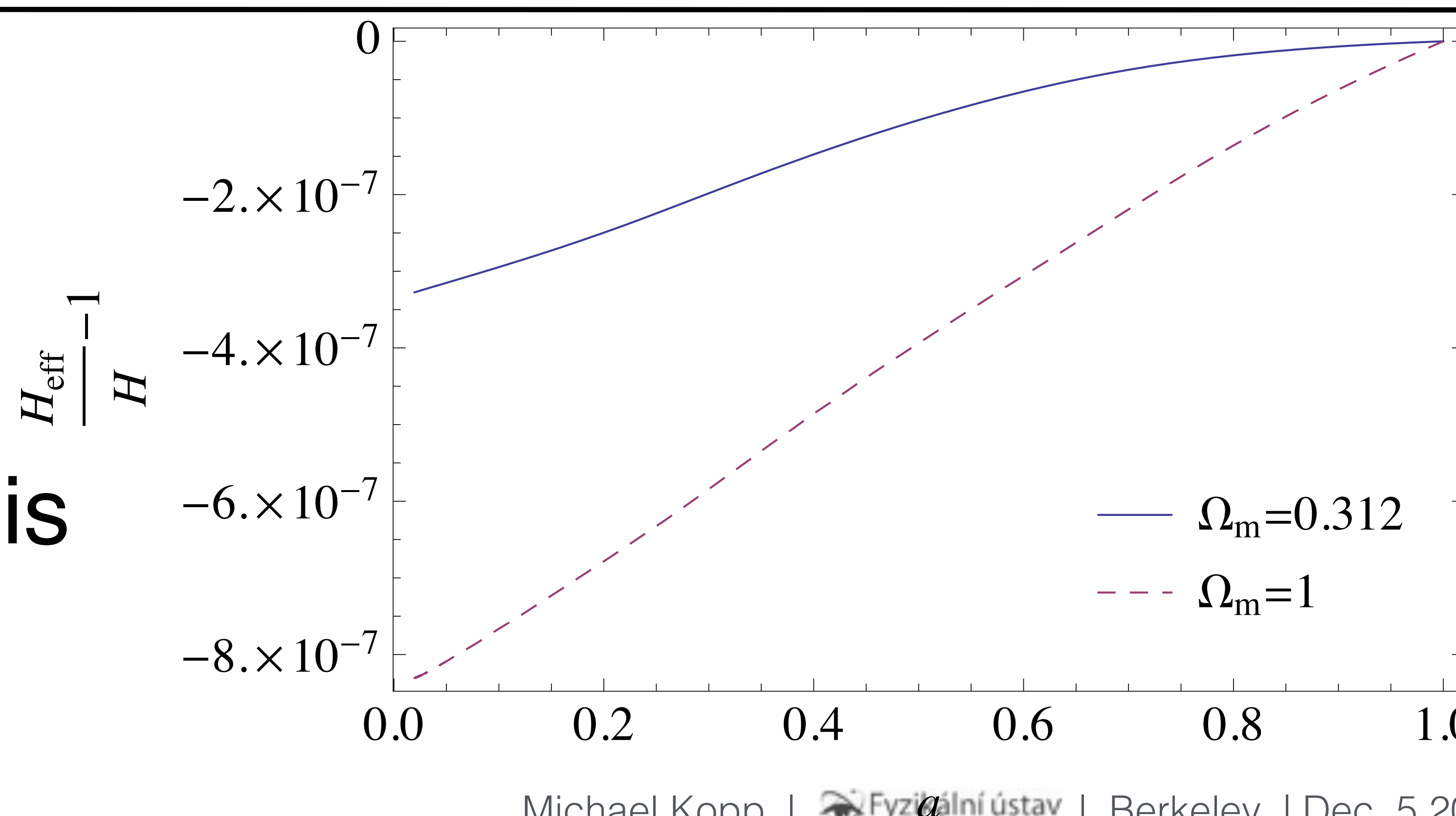
- Excellent agreement between ColDICE and ScM
- ScM can be used as basis for EFTofLSS Baumann, Nicolis, Senatore et al (JCAP, 2012)



Backreaction estimate

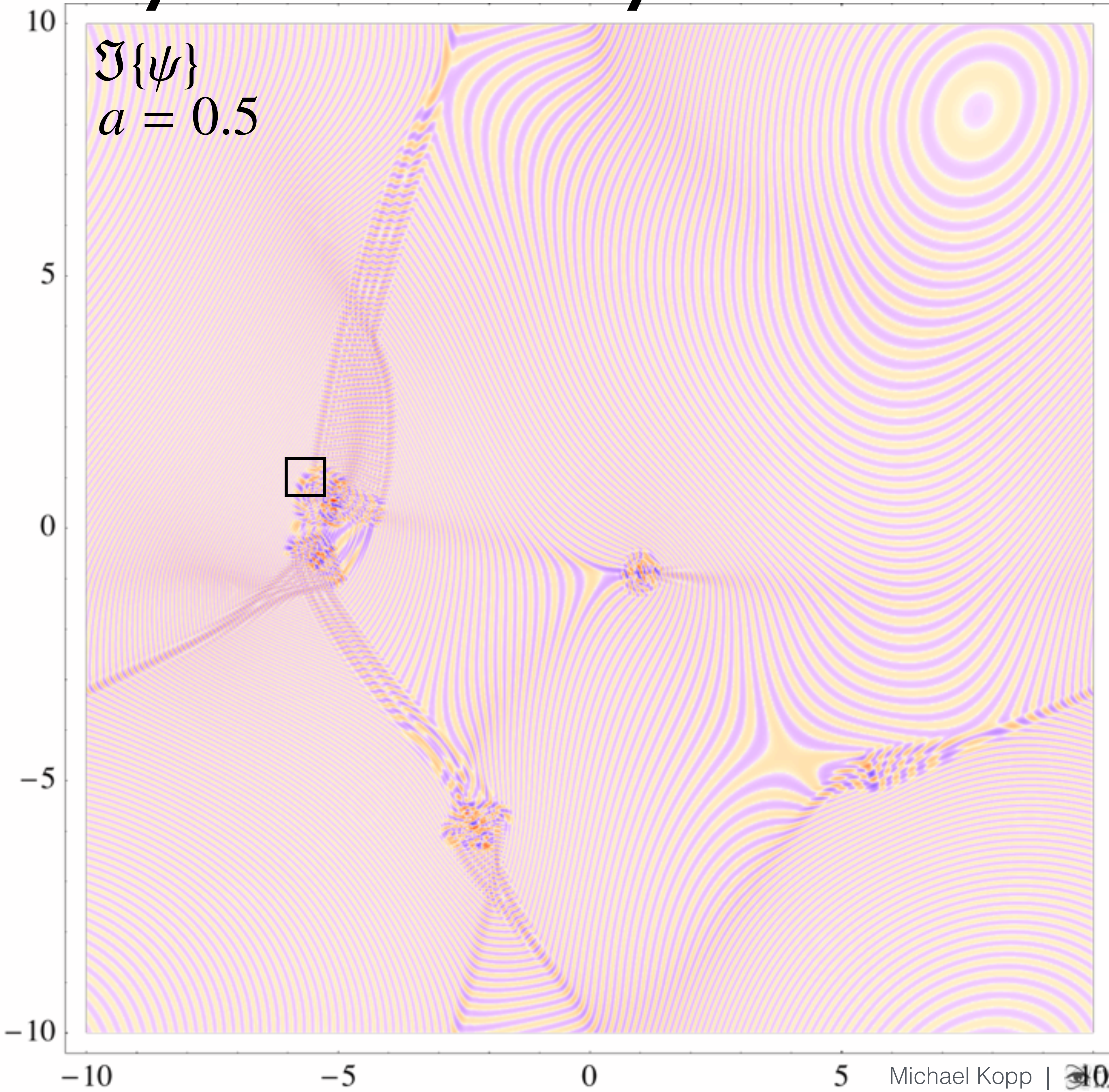
$$3H^2_{\rm eff} + 2\dot{H}_{\rm eff} = -8\pi G\langle P_{\rm eff}\rangle + \Lambda$$

- effect on expansion rate is negligibly small (in 2D)

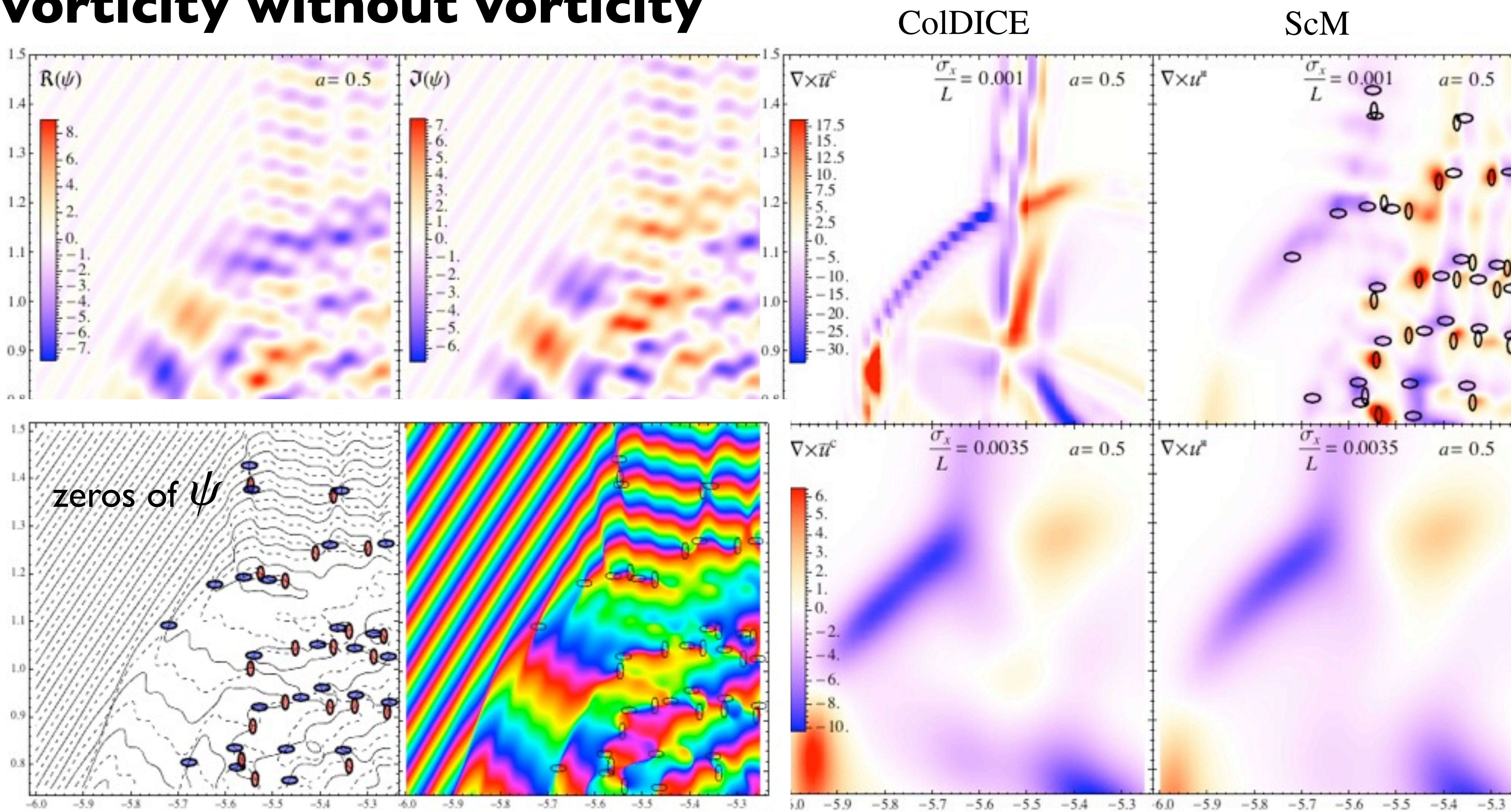


3b) Discussion:
Microscopic and macroscopic vorticity in CDM and ScM

vorticity without vorticity



vorticity without vorticity



$$\psi \text{ single valued} \Rightarrow \frac{1}{2\pi\tilde{\hbar}}\oint_C \nabla\phi\cdot d\boldsymbol{l} = m \in \mathbb{Z} \overset{\text{Stokes}}{\Rightarrow} \nabla\times(\nabla\phi) = \hat{\boldsymbol{z}}\,2\pi\,\tilde{\hbar}\,m\,\delta_{\mathrm{D}}(\boldsymbol{x}_{\mathrm{vort}})$$

$$\nabla\times\boldsymbol{u}_{\mathrm{H}} = \hat{\boldsymbol{z}}\,2\frac{\sigma_u}{\sigma_x}\sum_i^{N_{\mathrm{vort}}} m_i e^{\frac{(\boldsymbol{x}-\boldsymbol{x}_i)^2}{2\sigma_x{}^2}} + \sigma_x^2\left(\nabla\frac{n^{\mathrm{H}}_{,i}}{n^{\mathrm{H}}}\times\nabla\bar{\phi}_{,i}\right) + \mathcal{O}(\sigma_x{}^4)$$

Vorticity in CDM:
Pueblas, Scoccimarro
(PRD 2009)

4) Summary

MK, Vattis, Skordis, 1711.00140

1. Convergence of ScM to coarse grained Vlasov for $\tilde{\hbar} \to 0$
2. Excellent agreement with `ColDICE` Sousbie, Colombi 1509.07720
3. Many advantages: Uhlemann, MK, Haugg 1403.5567
   a) only 2 degrees of freedom, UV complete
   b) phase space can be avoided
   c) quasi-local in eulerian space
   d) *f* sampled uniformly, but minimal resolution $\tilde{\hbar}$

Future plans

1. 3D implementation with AMR, using `GAMER`

Schive et al (ApJS, 186, 2010)
Schive Chiueh, Broadhurst (Nature 2014)

2. Warm initial conditions, modeling neutrinos
3. Non-perturbative field theory methods applied to ScM

Backup slides

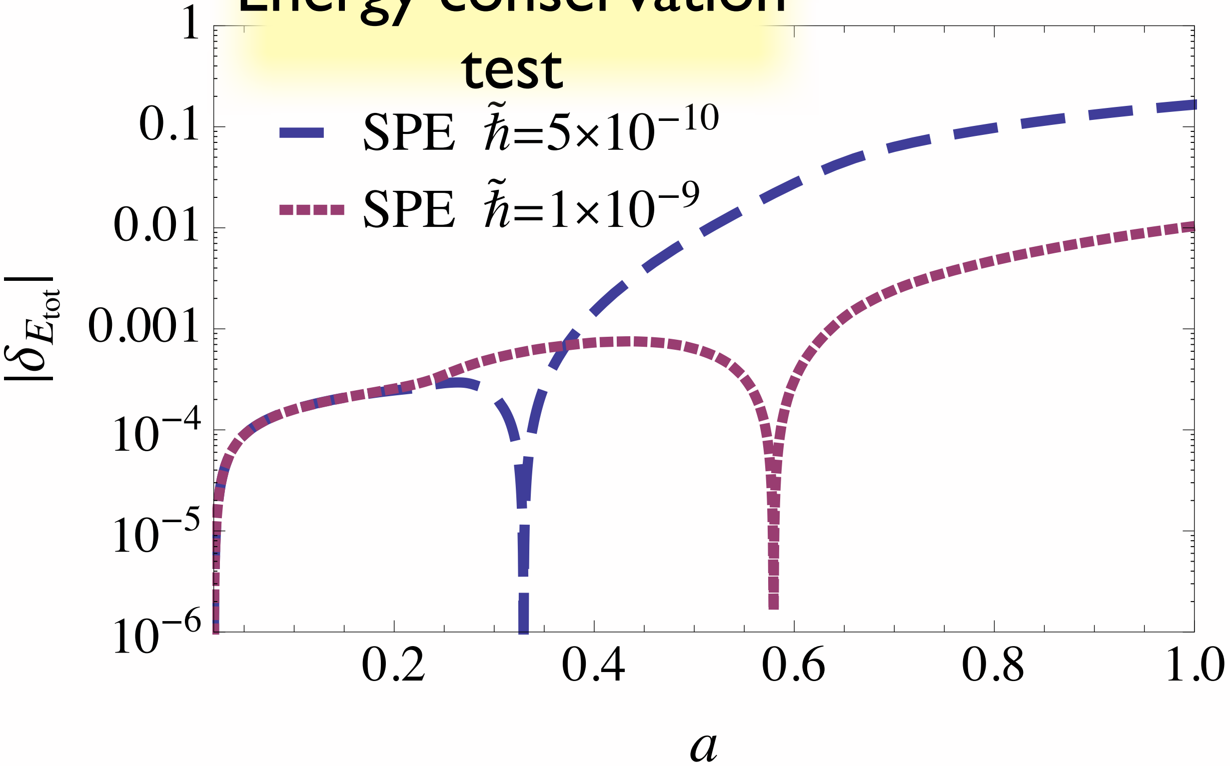
Gaussian random field collapse

Numerical convergence

- is easier for larger
- less than `ColDICE` (has AMR)

$$\delta_{E_{\rm tot}} \equiv \frac{E_{\rm tot}(a)}{E(a_{\rm ini})} - 1$$

Energy conservation test

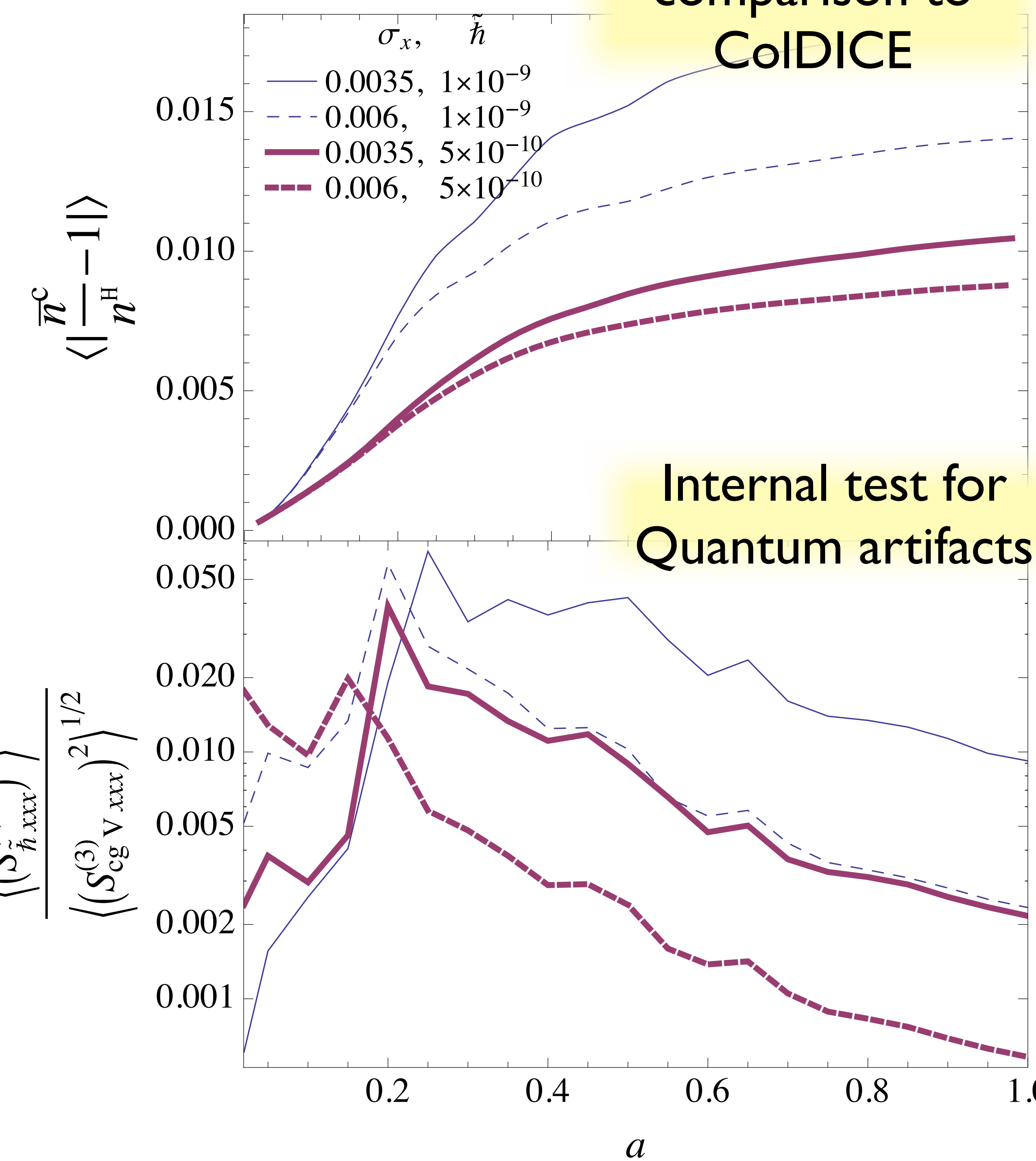


convergence to Vlasov

- is better for smaller $\tilde{\hbar}$
- $\langle \bar{n}^{\rm C}/n^{\rm H} - 1\rangle \lesssim 1\%$

comparison to ColDICE

Internal test for Quantum artifacts



Discretization and solution of Schrödinger Equation

$$i\tilde{\hbar}\partial_a\psi = -\frac{\tilde{\hbar}^2}{2a^3H}\Delta\psi + \frac{\Phi_\psi}{aH}\psi.$$

---

operator splitting via alternating direction implicit (ADI) method

Peaceman, Rachford (1955)
Guenther, (1995).

$\lambda = da/\epsilon$

$$e^{-\frac{i\lambda\tilde{\hbar}\epsilon}{4Ha^3}\frac{\partial^2}{\partial x^2}}S(x,y) = e^{\frac{i\lambda\tilde{\hbar}\epsilon}{4Ha^3}\frac{\partial^2}{\partial x^2}}\psi(a,x,y)$$

$$e^{-\frac{i\lambda\tilde{\hbar}\epsilon}{4Ha^3}\frac{\partial^2}{\partial y^2}}T(x,y) = e^{\frac{i\lambda\tilde{\hbar}\epsilon}{4Ha^3}\frac{\partial^2}{\partial y^2}}S(x,y)$$

$$e^{\frac{i\lambda\epsilon}{2\tilde{\hbar}Ha}\Phi_\psi}\psi(a+\lambda\epsilon,x,y) = e^{-\frac{i\lambda\epsilon}{2\tilde{\hbar}Ha}\Phi_\psi}T(x,y)$$

---

5 point stencil finite differences

$$\frac{\partial^2 f(x_0)}{\partial x^2} \approx \frac{1}{12\epsilon^2}\hat{\delta}_x f(x_0)$$

$$\hat{\delta}_x f(x_0) \equiv -f(x_0-2\epsilon) + 16f(x_0-\epsilon) - 30f(x_0) + 16f(x_0+\epsilon) - f(x_0+2\epsilon)$$

Crank-Nicolson form.
Pentadiagonal matrix inversion

Goldberg, Schey, Schwartz (1967)
Jia, Jiang (2013)

- unconditionally stable
- mass conserving
- error in time, space: $\mathcal{O}(\lambda^2\epsilon^2), \mathcal{O}(\epsilon^4)$

$$\left(1 - \frac{i\lambda\tilde{\hbar}}{48\epsilon Ha^3}\hat{\delta}_i\right)S_{ij} = \left(1 + \frac{i\lambda\tilde{\hbar}}{48\epsilon Ha^3}\hat{\delta}_i\right)\psi^n_{ij}$$

$$\left(1 - \frac{i\lambda\tilde{\hbar}}{48\epsilon Ha^3}\hat{\delta}_j\right)T_{ij} = \left(1 + \frac{i\lambda\tilde{\hbar}}{48\epsilon Ha^3}\hat{\delta}_j\right)S_{ij}$$

$$\left(1 + \frac{i\lambda\epsilon}{2\tilde{\hbar}Ha}\Phi^n_{\psi\,ij}\right)\psi^{n+1}_{ij} = \left(1 - \frac{i\lambda\epsilon}{2\tilde{\hbar}Ha}\Phi^n_{\psi\,ij}\right)T_{ij}$$

Initial conditions

$\phi_P(\boldsymbol{q})$ here: gaussian random field, or product of sines

$$\boldsymbol{P}(\boldsymbol{q}) = \nabla_q \phi_P(\boldsymbol{q})$$

$$\boldsymbol{\Psi} = D(a)\, \boldsymbol{P}(\boldsymbol{q})$$

displacement field in the Zel'dovich appr.

initial velocities and positions

$$\boldsymbol{u}_{\rm d}(\boldsymbol{X}) = \partial_\eta\big|_q \boldsymbol{\Psi}(\boldsymbol{q}),$$

$$\boldsymbol{x} = \boldsymbol{X}(t,\boldsymbol{q}) = \boldsymbol{q} + \boldsymbol{\Psi}(t,\boldsymbol{q}),$$

Eulerian density and velocity potential of a dust fluid

$$n_{\rm d}(t,\boldsymbol{x}) = \left\{\det\left[\delta_{ij} + D(a)\partial_{q^i}\partial_{q^j}\phi_P(\boldsymbol{q})\right]\right\}^{-1}\Big|_{\boldsymbol{q}=\boldsymbol{Q}(t,\boldsymbol{x})}$$

$$\phi_{\rm d}(t,\boldsymbol{x}) = a^2 H f D\left(\phi_P(\boldsymbol{q}) + \frac{1}{2}D(a)\,|\boldsymbol{P}(\boldsymbol{q})|^2\right)\Big|_{\boldsymbol{q}=\boldsymbol{Q}(t,\boldsymbol{x})},$$

Initial wave function

$$\psi_{\rm ini}(\boldsymbol{x}) = \sqrt{n_{\rm d}^{\rm ini}}\exp(i\phi_{\rm d}^{\rm ini}/\tilde{\hbar})$$

Computation with a single GPU

- Nvidia K20X GPU (Kepler architecture)
- CUDA C
- CuFFT
- 6GB memory
- ⇒8192^2 maximum resolution

Comparison: ScM dust CDM

	ScM: $f_{\mathrm{H}}(t, \boldsymbol{x}, \boldsymbol{u})$	Dust: $f_{\mathrm{d}}(t, \boldsymbol{x}, \boldsymbol{u})$	CDM: $f_{\mathrm{c}}(t, \boldsymbol{x}, \boldsymbol{u})$
Degrees of freedom (d.o.f.) type	$1{\times}\mathbb{C}$: $\psi(t, \boldsymbol{x})$	$2{\times}\mathbb{R}$: $n_{\mathrm{d}}(t, \boldsymbol{x})$, $\phi_{\mathrm{d}}(t, \boldsymbol{x})$	$2 \times \mathbb{R}^d$: $\boldsymbol{X}(t, \boldsymbol{q})$, $\boldsymbol{U}(t, \boldsymbol{q})$
Number of d.o.f.	2	2	$2d$
Equations of motion	1st order SPE	1st order fluid equations	1st order trajectories
Singularity-free d.o.fs	✓	✗	✓
$f(\boldsymbol{x}, \boldsymbol{u})$ constructed from	$\psi(\boldsymbol{x})$, quasi-local	$n_{\mathrm{d}}(\boldsymbol{x})$, $\boldsymbol{\nabla}\phi_{\mathrm{d}}(\boldsymbol{x})$, local	$\boldsymbol{X}(\boldsymbol{q}), \boldsymbol{U}(\boldsymbol{q})$, non-local
$M^{(n)}(\boldsymbol{x})$ constructed from	$\partial_x^{0\leq m\leq n}\psi(\boldsymbol{x})$, quasi-local	$n_{\mathrm{d}}(\boldsymbol{x})$, $\boldsymbol{\nabla}\phi_{\mathrm{d}}(\boldsymbol{x})$, local	$\boldsymbol{X}(\boldsymbol{q}), \boldsymbol{U}(\boldsymbol{q})$, non-local
Vlasov equation solved	approximately $(\hbar, \sigma_x)$	exactly until shell-crossing	exactly
Multi-streaming and virialization	✓	✗	✓
Initial conditions type	arbitrary, incl. cold	cold	cold