The Effect of Gravitational Lensing on Cosmological Parameter Estimation

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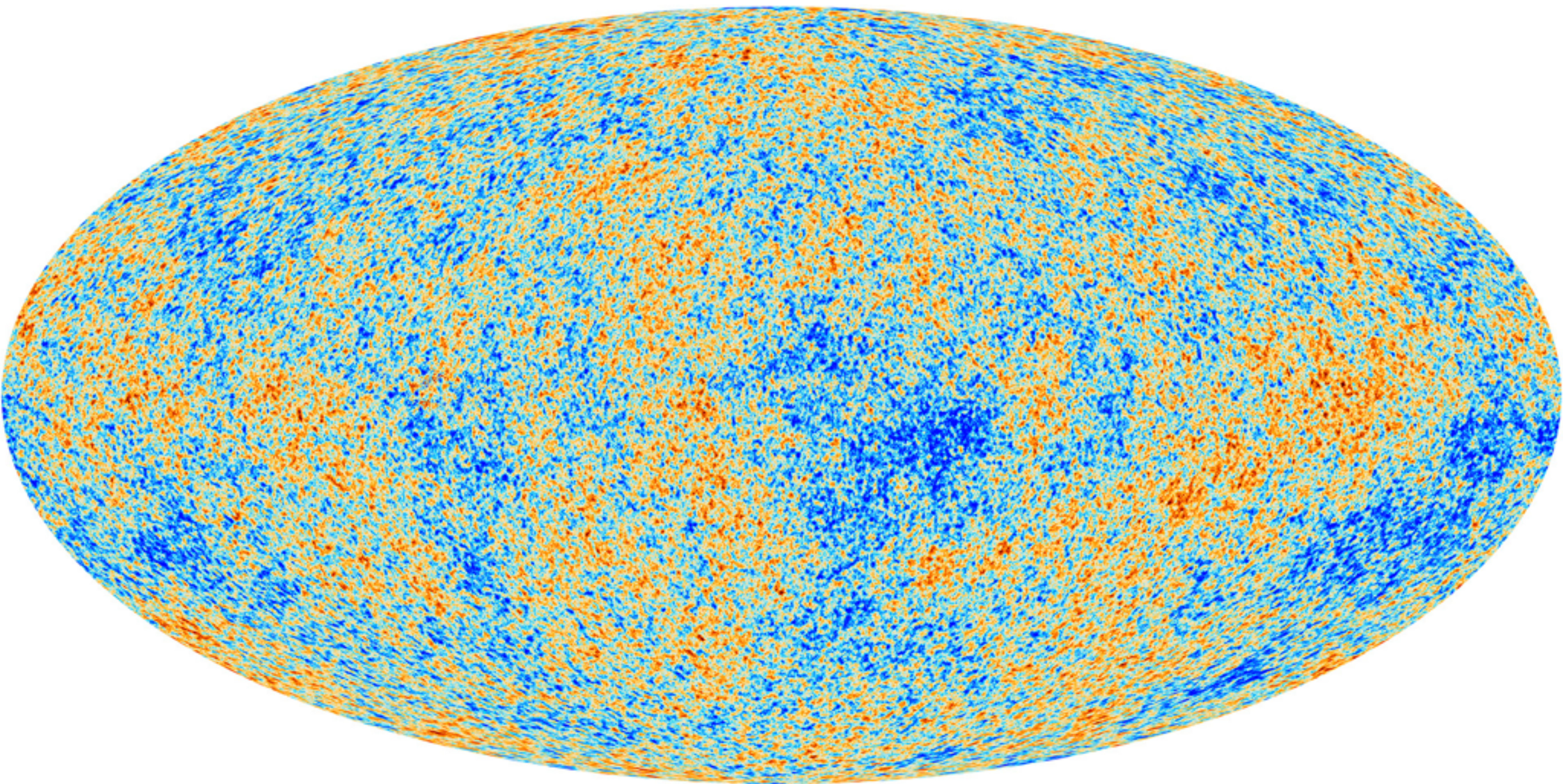
LBL Seminar, 2016/09/16

Context: cosmological parameters from the CMB

It is usually assumed that we are looking here at a spherical surface at $z \sim 1100$ with $D = D_0(z=1100)$

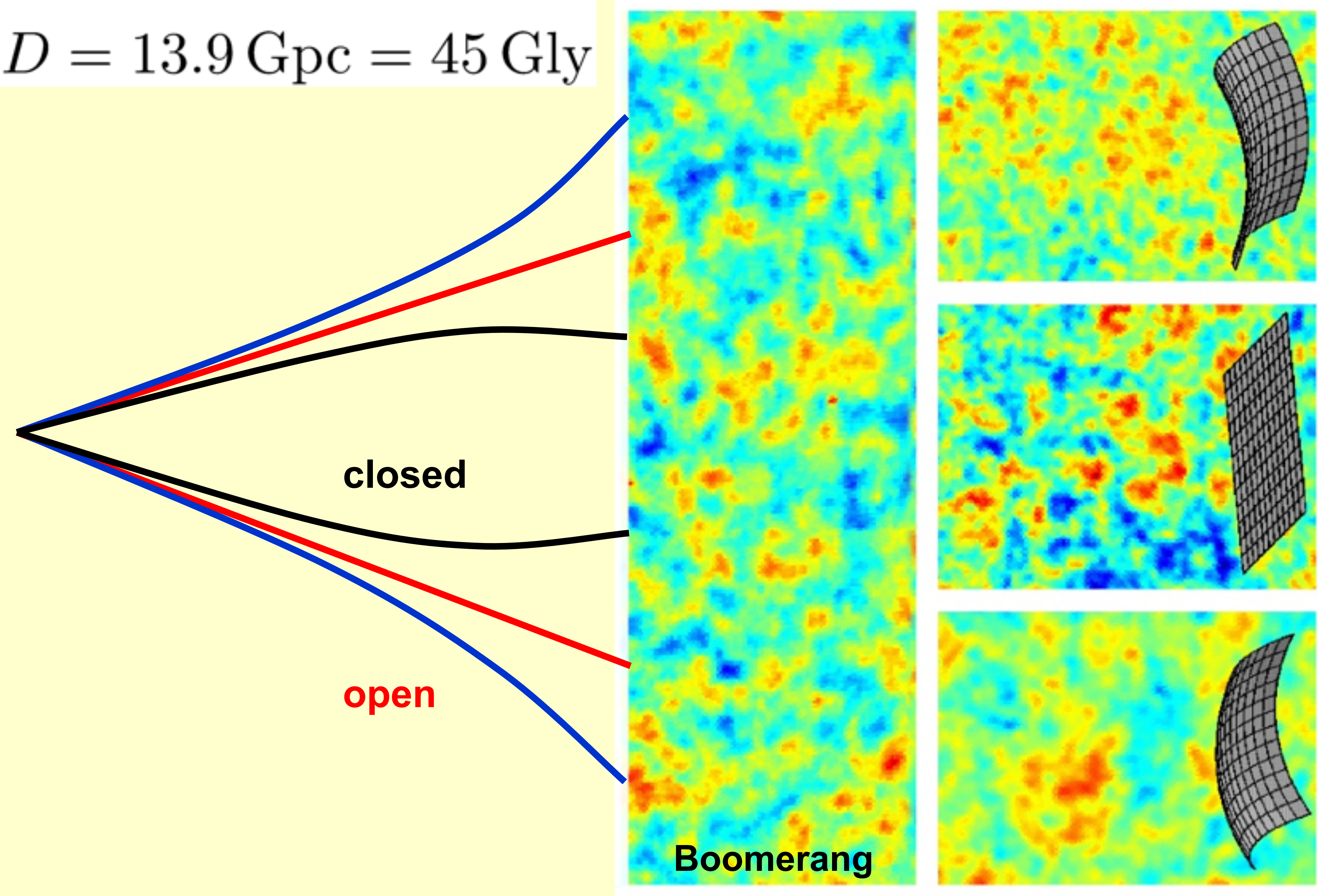
But are we?



How far away is the CMB?

$$D = \int \frac{c}{H(z)}\, dz$$

$$z = 1080 \Rightarrow D = 13.9\,\mathrm{Gpc} = 45\,\mathrm{Gly}$$






What is the distance to the CMB?
How relativistic corrections remove the tension with local H_0 measurements

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The success of precision cosmology depends not only on accurate observations, but also on the theoretical model – which must be understood to at least the same level of precision. Subtle relativistic effects can lead to biased measurements if they are neglected. One such effect gives a systematic shift in the distance-redshift relation away from its background value, due to the accumulation of all possible lensing events. We estimate the expectation value of this aggregated lensing using second-order perturbations about a concordance background, and show that the distance to last scattering is shifted by several percent. Neglecting this shift leads to significant bias in the background cosmological parameters. We show that this removes the tension between local measurements of H_0 and those measured through the CMB and favours a closed universe.

CUMD14: few% perturbation to CMB area from lensing!

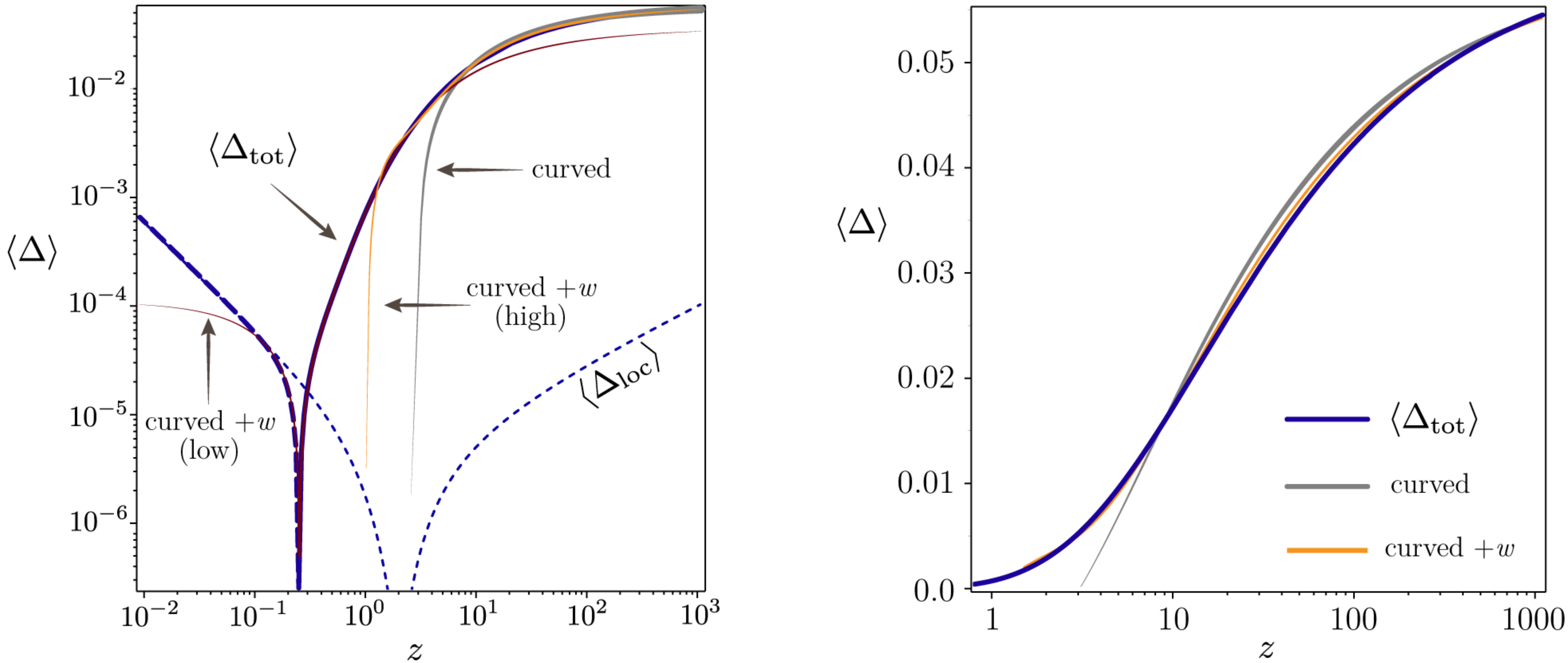
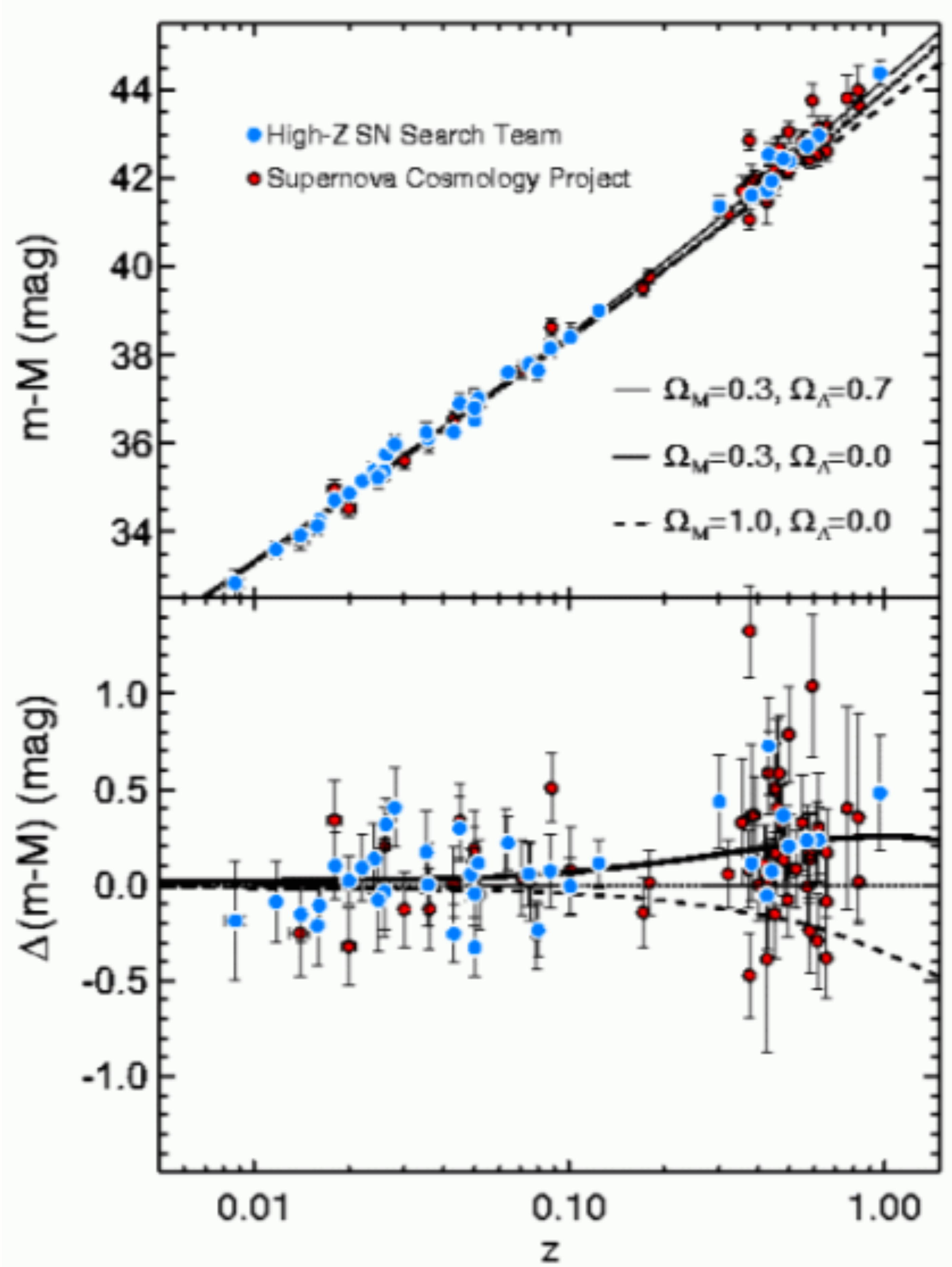


Figure 1: Fractional correction $\langle\Delta\rangle(z)$ to the distance [see (3)] for a fiducial model $\Omega_m = 0.35, h = 0.65, w = -1$ and $n_s = 1$, showing local (dotted) and total (solid) $\langle\Delta\rangle$ (left). The correction is negative for $z \lesssim 0.25$ (dashed), purely from the local contribution (dotted). At high $z \gtrsim 10$ the corrections are similar to an open ΛCDM model with $\Omega_K^{\rm eff} \approx 0.043$ (grey 'curved', shown for high z). An open model with evolving dark energy is a better fit down to $z \sim 3$ [orange, 'curved+w (high)']. For low z an effective open model with percent level changes to the background parameters gives a good approximation to $\langle\Delta\rangle(z)$ [red, 'curved+w (low)']. Right, we show $\langle\Delta\rangle$ together with two approximations discussed later in the text [see (20) and below], illustrating the accumulation of lensing at high redshift.

Few percent changes to cosmological parameter from CMB

If correct this would have implications for SN1a cosmology too

Hubble diagram from SN1a - assumes no flux *bias* from lensing

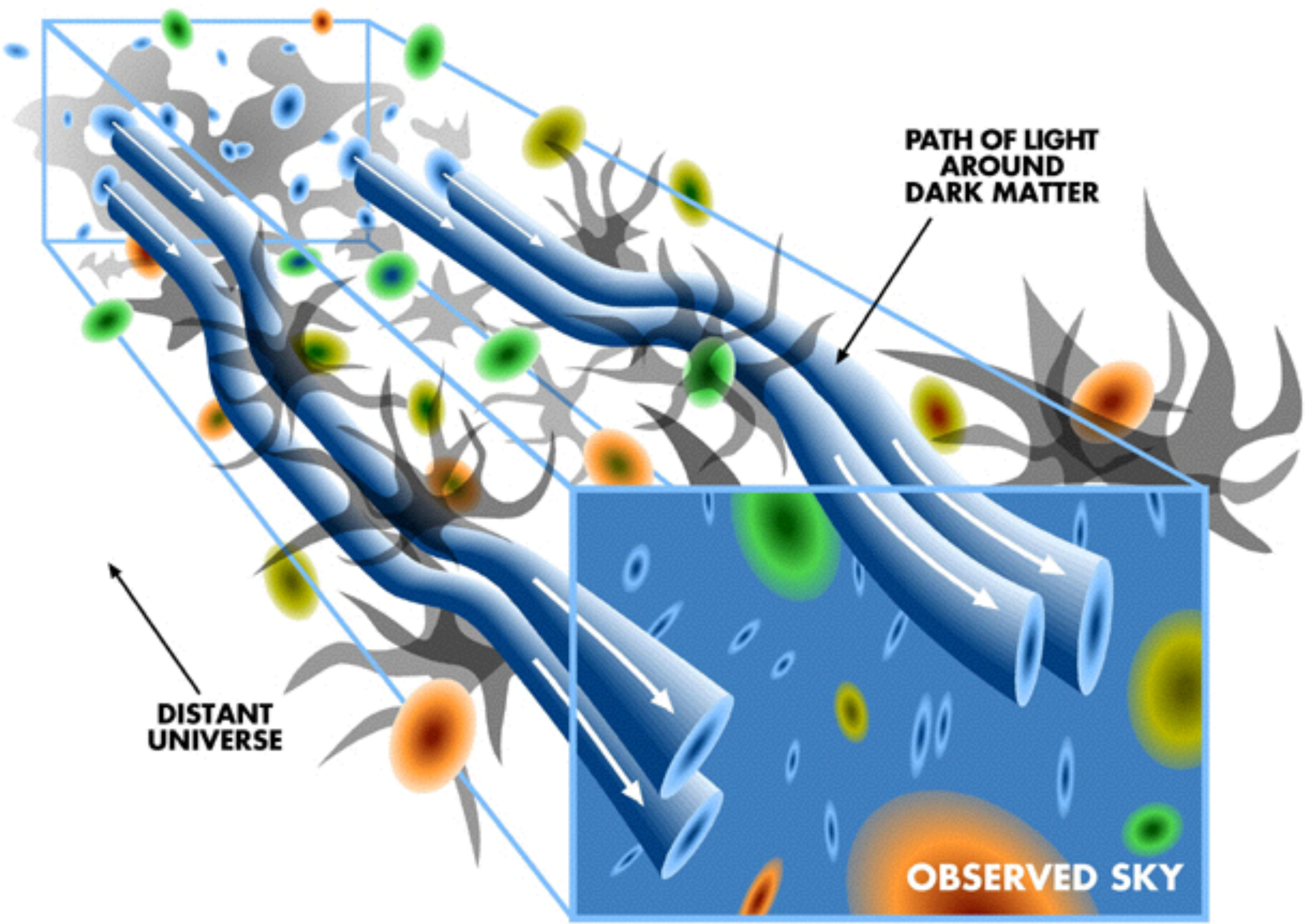


Outline of talk

- **Some preliminaries**
  - what do we mean by *distance* in cosmology?
  - basics of gravitational lensing - light deflection, shear & magnification
- **Historical review:**
  - Zel'dovich '63 .... Feynman & Gunn .... Kantowski ... Dyer & Roeder
  - Weinberg '76 - *no effect* for transparent lenses (flux conservation)
  - Schneider et al. ('84..'94): *magnification* and *focusing theorems*
    - based on Raychaudhuri, Sachs, ...
  - Metcalf and Silk '97: ~no mean magnification of the CMB
  - Ellis, Bassett & Dunsby '97 - critique of Weinberg '67
  - Kibble & Lieu '05 - distinguished *source* and *direction* averages
  - 2nd order cosmological perturbation theory (Umeh++; Marozzi++)
    - Clarkson, Ellis++ '12 - large ($O(\kappa^2)$) source magnification
    - Clarkson++ '14 - large ($O(\kappa^2)$) δ(photosphere area)
- **NK + John Peacock** arXiv:1503.08506:
  - 1) *reconcile the above, apparently contradictory, results*
  - 2) *Weinberg's argument contains a loophole - but it is very small*

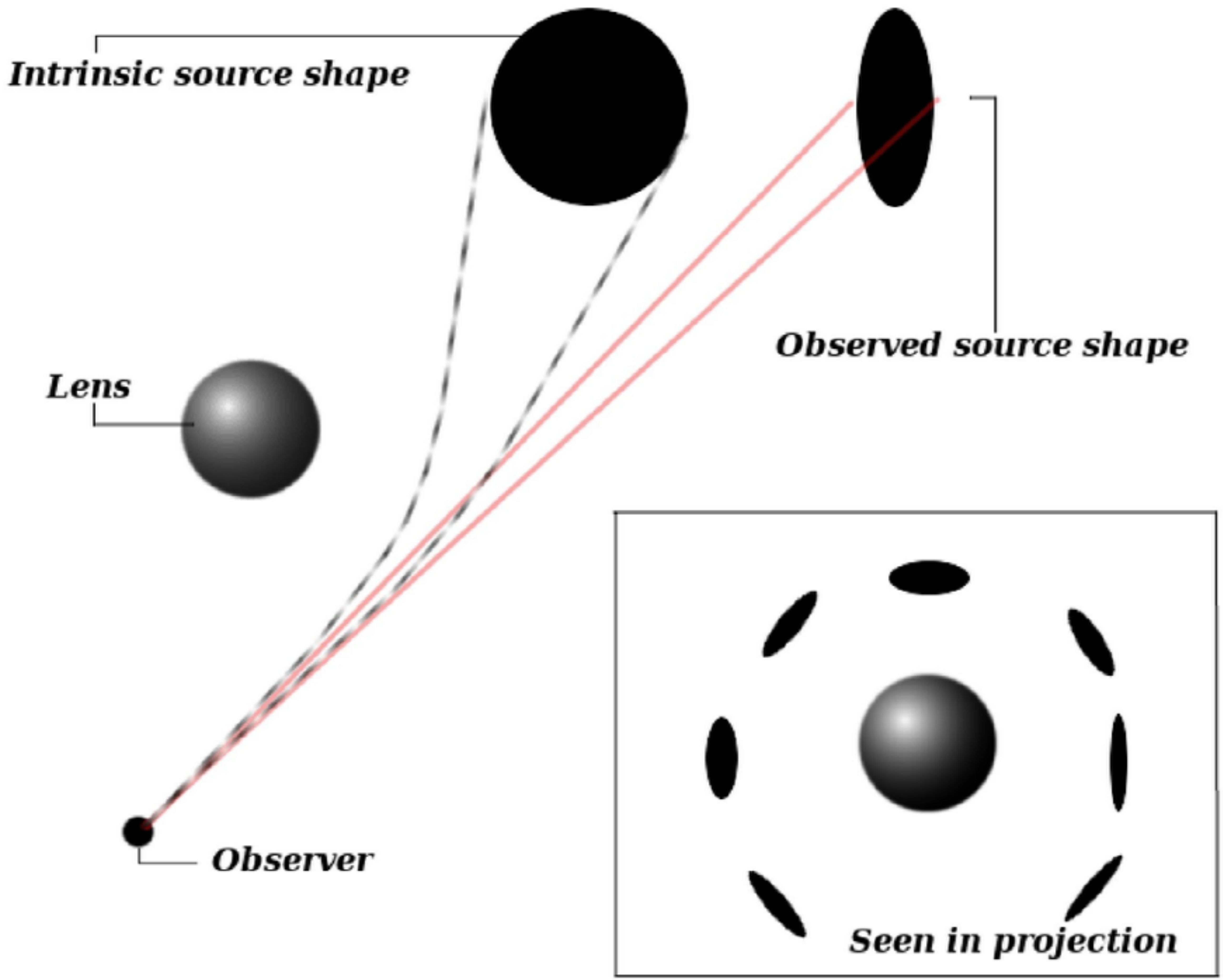
Preliminaries 1: What do we mean by "*distance*" in cosmology

- There are lots of ways to directly measure distances in astronomy
  - rulers (in principle)
  - parallaxes
  - radar echoes
- None of these are of much use in cosmology. Instead we have:
  - *redshift* (reflects change in size of the Universe)
  - `*conformal*' or `*comoving*' distance χ
    - appears in *spacetime metric* $ds^2 = -d\tau^2 + a^2(\tau)(d\chi + S_k^2(\chi)\, d\sigma^2)$
    - not observable, but useful to relate other observable
  - **angular diameter distance**: $dl = a(\tau(\chi))\, S_k(\chi)\, d\theta = D_A\, d\theta$
  - **luminosity distance:** $F = L / (4 \pi D_L^2)$
    - these are both **"apparent"** distances
    - require standard "candles" or "measuring rods"
- Here we are interested in D_A and D_L:
  - Lensing changes D_A, D_L: they become random functions of direction
  - key question: **does structure *bias* angular sizes or flux densities?**


PATH OF LIGHT
AROUND
DARK MATTER
DISTANT
UNIVERSE
OBSERVED SKY

Preliminaries 2: basic of grav. lensing: deflection & shear

- Basic quantities in gravitational lensing
  - Gravitational *time delay* (Shapiro): $\Delta t = 2 \int d\lambda\ \Phi/c^2$
    - λ = distance: Φ = gravitational field from $\Delta\varrho/\varrho$
    - measured in solar system
    - and in "strong lensing" - multiple images of quasars
  - Light deflection $\theta_1 \sim \int d\lambda \nabla\Phi/c^2 \sim GM/bc^2 \sim (H\lambda/c)^2\Delta$
    - cumulative deflection is a "random walk"
      - $\theta \sim N^{1/2}\ \theta_1 \sim (H\lambda/c)^{3/2}\Delta$
      - $\Delta = \Delta\varrho/\varrho \sim \xi^{1/2} \sim 1/\lambda$
    - so θ dominated by large scale structure (~30 Mpc)
      - quite large ~ few arc-minutes ~ 10^{-3} radians at high z
      - but not directly observable


Intrinsic source shape
Observed source shape
Lens
Observer
Seen in projection

Preliminaries 2: basic of grav. lensing: deflection & shear

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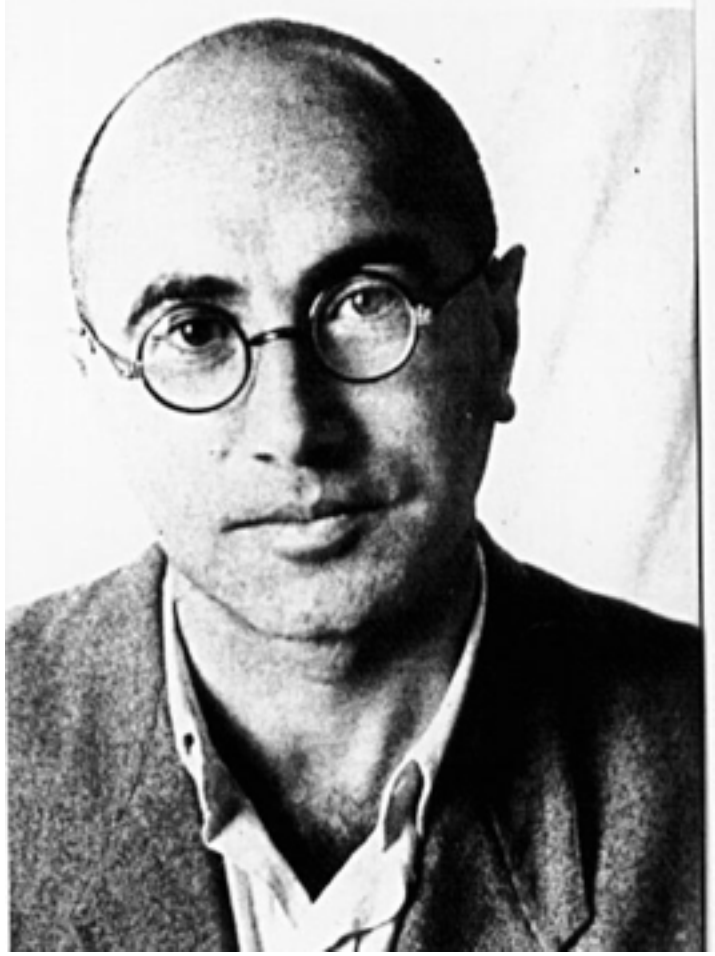
      - but not directly observable
  - Observable in WL is the *gradient* of the deflection angle
    - described by a 2x2 image distortion tensor
      - trace: $\varkappa$ (kappa) → *magnification* (changes *size* of objects)
      - 2 other components: γ (gamma) → *image shear* (changes *shapes*)
    - $\varkappa$, $\gamma \sim 10^{-2}$ at ~ degree scales for sources at z ~ 1
    - $\varkappa^2$, $\gamma^2 \sim 10^{-3}$ at ~ degree scales for sources at z >> 1 (e.g. CMB)
    - but scales with size of structure as $\sim \lambda^{-1}$
      - so potentially *very large effects* from small-scale structures



OBSERVATIONS IN A UNIVERSE HOMOGENEOUS IN THE MEAN

Ya. B. Zel'dovich

Translated from Astronomicheskii Zhurnal, Vol. 41, No. 1,
pp. 19-24, January-February, 1964
Original article submitted June 12, 1963

A local nonuniformity of density due to the concentration of matter of the universe into separate galaxies produces a significant change in the angular dimensions and luminosity of distant objects as compared to the formulas for the Friedman model.

The propagation of light in a homogeneous and isotropic model of the expanding universe (first studied by A. A. Friedman) has been investigated in a number of papers [1, 2, 3].

In these papers expressions were obtained for the observed angular diameter Θ and the observed brightness of an object with a known absolute diameter and absolute brightness as a function of the distance or, strictly speaking, the red shift of the object $\Delta = (\omega_0 - \omega)/\omega_0$.

In particular, there is a remarkable feature in the function $\Theta(\Delta)$, namely, the presence of a minimum when Δ is approximately equal to 1/2. Formula (10) and Fig. 6 in the appendix show the variation of the function $f(\Delta) = rH/c\Theta$ which is inversely proportional to Θ for a given density of matter. Here $\underline{r}$ is the radius of the object, H is Hubble's

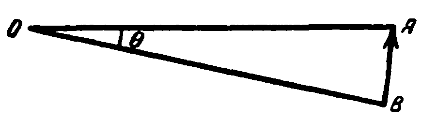


Fig. 1.

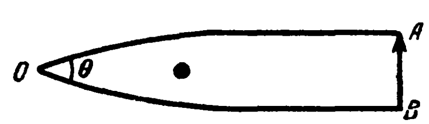
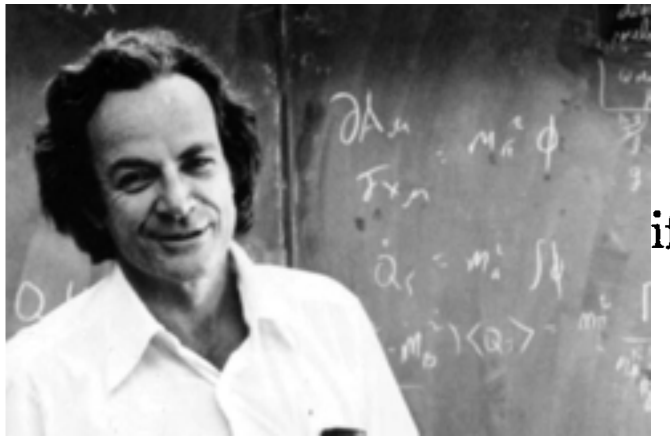


Fig. 2.

A mass situated between these rays bends the latter in such a way that Θ is increased (Fig. 2). What we have in mind is the bending of light rays by the gravitational field predicted by Einstein; this bending amounts to 1.75" for a light ray passing near the limb of the solar disc and has been confirmed by observation.

ON THE PROPAGATION OF LIGHT IN INHOMOGENEOUS COSMOLOGIES. I. MEAN EFFECTS



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Received February 23, 1967; revised May 23, 1967

ABSTRACT

The statistical effects of local inhomogeneities on the propagation of light are investigated, and deviations (including rms fluctuations) from the idealized behavior in homogeneous universes are investigated by a perturbation-theoretic approach. The effect discussed by Feynman and recently by Bertotti of the density of the intergalactic medium being systematically lower than the mean mass density is examined, and expressions for the effect valid at all redshifts are derived.

I. INTRODUCTION

In an unpublished colloquium given at the California Institute of Technology in 1964, Feynman discussed the effect on observed angular diameters of distant objects if the intergalactic medium has lower density than the mean mass density, as would be the case if a significant fraction of the total mass were contained in galaxies. It is an obvious extension of the existence of this effect that luminosities will also be affected, though this was apparently not realized at the time. This realization prompted the conviction that the effect of known kinds of deviations of the real Universe from the homogeneous isotropic models (upon which predictions had been based in the past) upon observable quantities like luminosity and angular diameter should be investigated. The author (1967) has recently made such a study for angular diameters; the present work deals primarily with mean statistical effects upon luminosity. A third paper will deal with possible extreme effects one may expect to encounter more rarely. Some of the results discussed here have been discussed independently by Bertotti (1966) and Zel'dovich (1965).

Kantowski '69

CORRECTIONS IN THE LUMINOSITY-REDSHIFT RELATIONS OF THE HOMOGENEOUS FRIEDMANN MODELS

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Received January 22, 1968; revised March 22, 1968

ABSTRACT

In this paper the bolometric luminosity-redshift relations of the Friedmann dust universes ($\Lambda = 0$) are corrected for the presence of inhomogeneities. The "locally" inhomogeneous Swiss-cheese models are used, and it is first shown that the introduction of clumps of matter into Friedmann models does not significantly affect the $R(z)$ or $R(v)$ relations (Friedmann radius versus the redshift or affine parameter) along a null ray. Then, by the use of the optical scalar equations, a linear third-order differential equation is arrived at for the mean cross-sectional area of a light beam as a function of the affine parameter. This differential equation is confirmed by rederiving its small redshift solution from an interesting geometrical point of view. The geometrical argument is then extended to show that "mild" inhomogeneities of a transparent type have no effect on the mean area of a light beam.

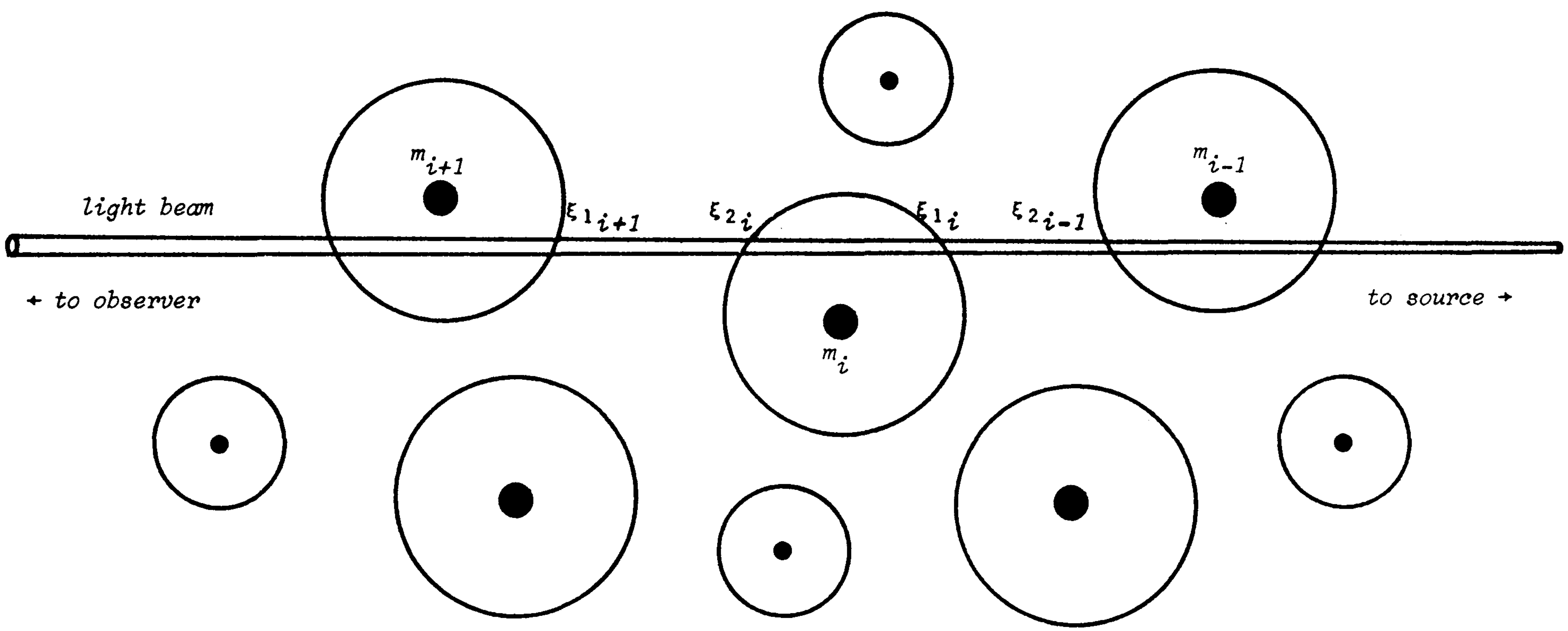

FIG. 1.—Spacelike section of a typical Swiss-cheese universe

Dyer & Roeder '72

THE DISTANCE-REDSHIFT RELATION FOR UNIVERSES WITH NO INTERGALACTIC MEDIUM

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Received 1972 April 19

ABSTRACT

The distance-redshift relation is derived for model universes in which there is negligible intergalactic matter and in which the line of sight to a distant object does not pass close to intervening galaxies. When fitted to observations, this relation yields a higher value of q_0 than does a homogeneous model.

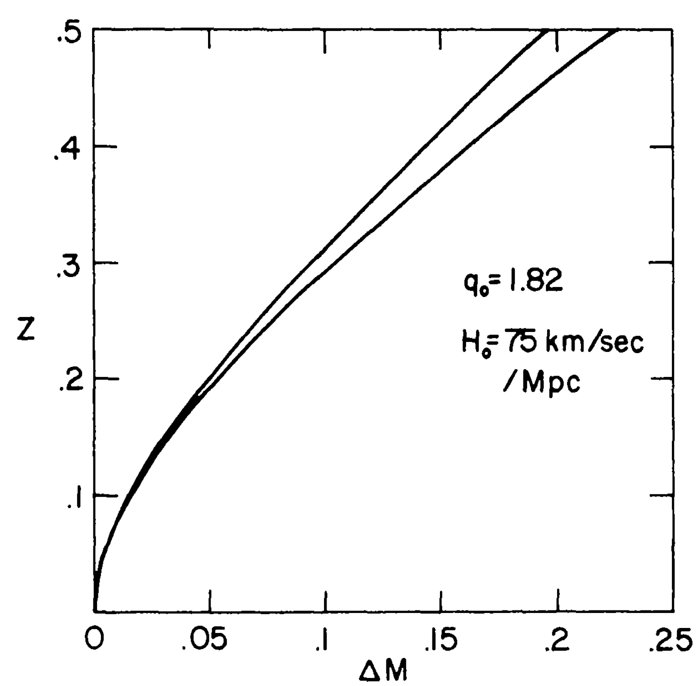

Fig. 1.—The dimming, relative to the homogeneous model, assuming that the beam passes far from any intervening galaxies (*lower curve*) and assuming that the beam passes no closer than 2 kpc to the center of galaxies similar to our own (*upper curve*).

Weinberg 1976 - no effect (flux conservation)

APPARENT LUMINOSITIES IN A LOCALLY INHOMOGENEOUS UNIVERSE

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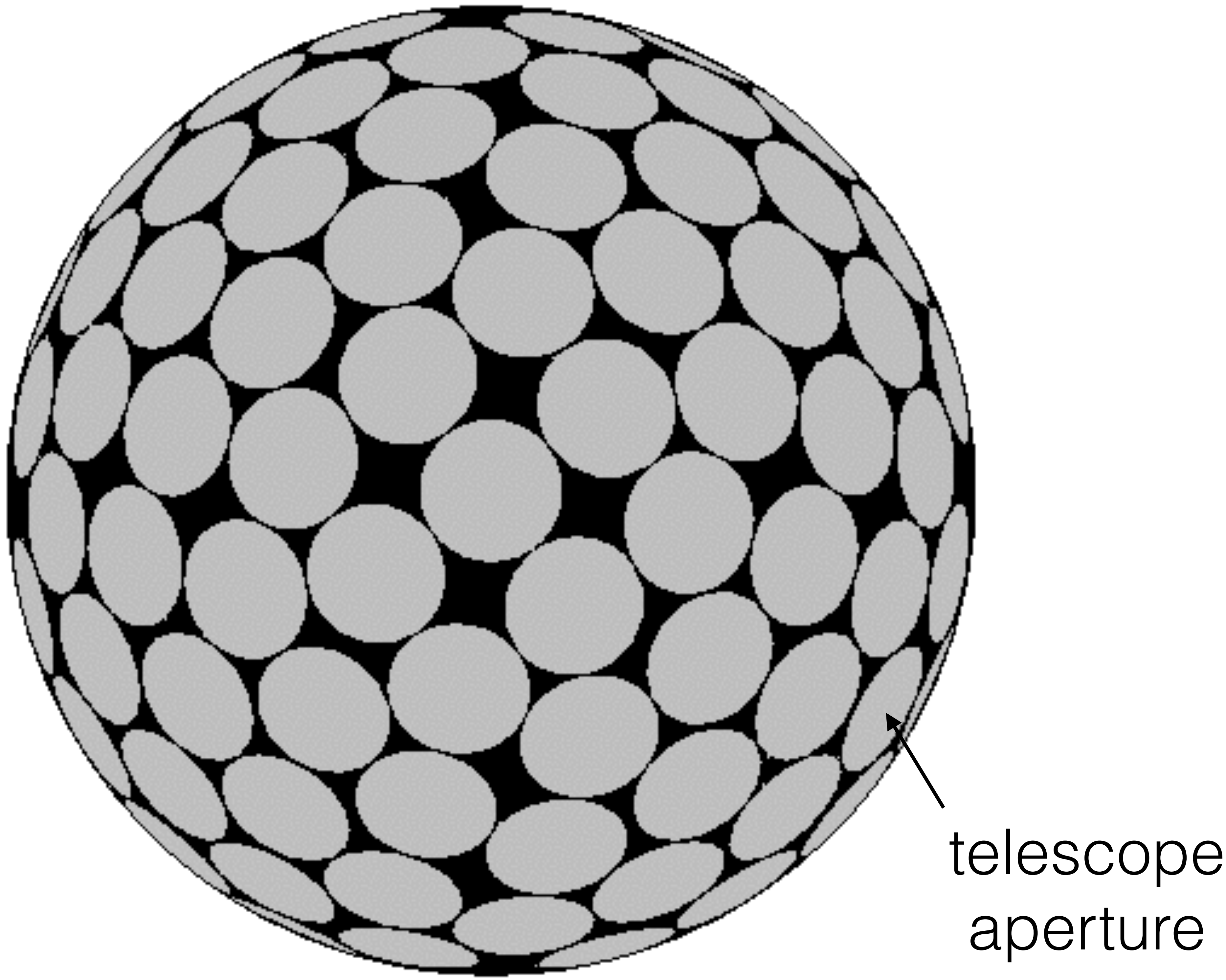
Received 1976 April 6; revised 1976 May 20

ABSTRACT

Apparent luminosities are considered in a locally inhomogeneous universe, with gravitational deflection by individual clumps of matter taken into account. It is shown that as long as the clump radii are sufficiently small, gravitational deflection by the clumps will produce the same average effect as would be produced if the mass were spread out homogeneously. The conventional formulae for luminosity distance as a function of redshift consequently remain valid, despite the presence of any local inhomogeneities of less than galactic dimensions. For clumps of galactic size, the validity of the conventional formulae depends on the selection procedure used and the redshift of the object studied.

Subject headings: cosmology — galaxies: redshifts — gravitation

Weinberg's argument (that <magnification> = 1)



But this *assumes* that the total area is unchanged

Lensing and caustic effects on cosmological distances.

EBD '98

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December 4, 2013

Abstract

We consider the changes which occur in cosmological distances due to the combined effects of some null geodesics passing through low-density regions while others pass through lensing-induced caustics. This combination of effects increases observed areas corresponding to a given solid angle even when averaged over large angular scales, through the additive effect of increases on all scales, but particularly on micro-angular scales; however angular sizes will not be significantly effected on large angular scales (when caustics occur, area distances and angular-diameter distances no longer coincide). We compare our results with other works on lensing, which claim there is no such effect, and explain why the effect will indeed occur in the (realistic) situation where caustics due to lensing are significant. Whether or not the effect is significant for number counts depends on the associated angular scales and on the distribution of inhomogeneities in the universe. It could also possibly affect the spectrum of CBR anisotropies on small angular scales, indeed caustics can induce a non-Gaussian signature into the CMB at small scales and lead to stronger mixing of anisotropies than occurs in weak lensing.

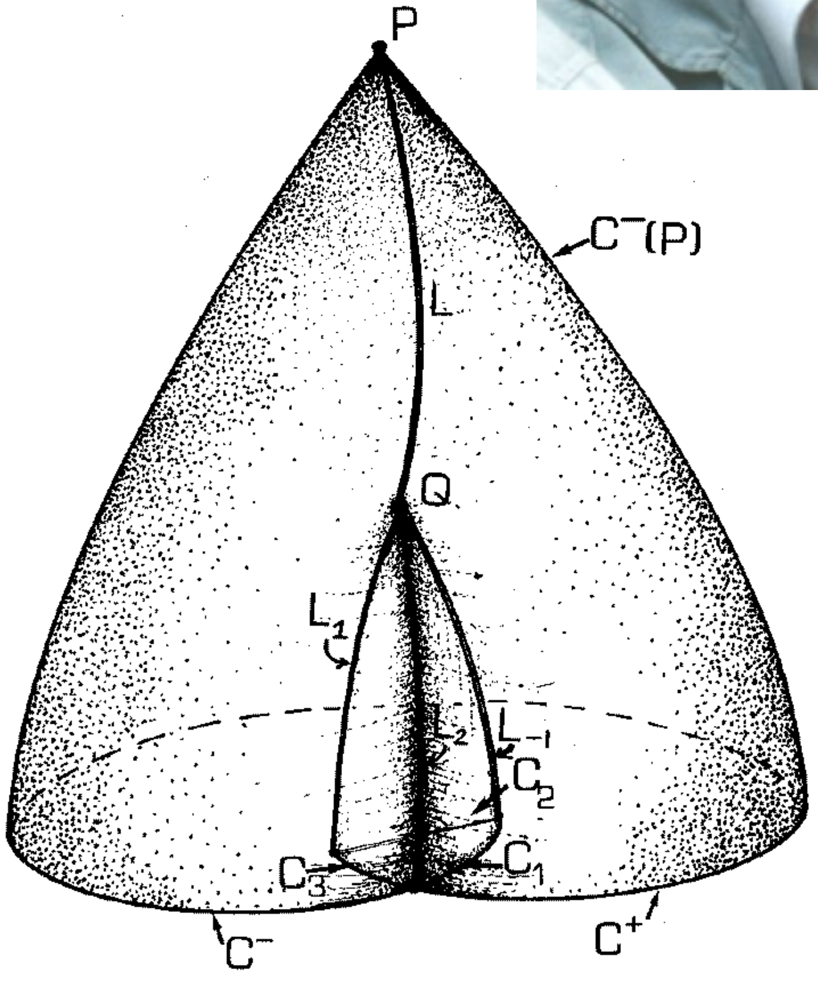
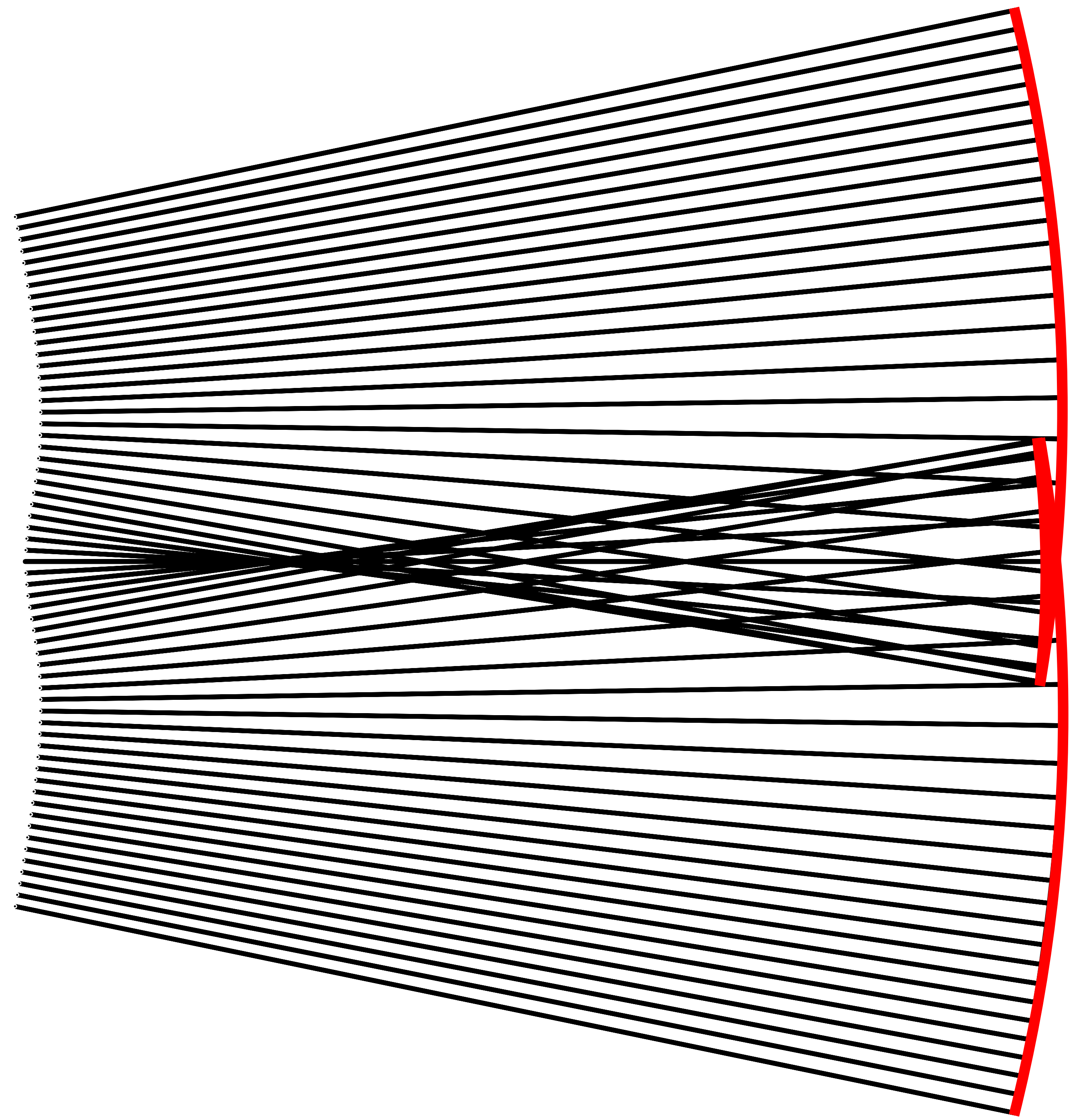


Figure 1: A lens L and resulting caustics on the past light cone $C^-(P)$ (2-dimensional section of the full light cone), showing in particular the cross-over line L_2 and cusp lines L_{-1}, L_1 meeting at the conjugate point Q. The intersection of the past light cone with a surface of constant time defines exterior segments C^-, C^+ of the light cone together with interior segments C_1, C_2, C_3.

Ellis, Bassett & Dunsby '98 critique of Weinberg '76

- EDB98 make two points:
- Weinberg *assumes* that which is to be proven
  - we agree: W76 assumes that the surface of constant z around a source (or observer) is a sphere
- Small scale strong lensing causes the surface to be folded over on itself so total area greatly enhanced
  - quite possibly true
- Thus Weinberg's claim is disproved
  - we disagree: W76 still applies if multiple images are unresolved



Enter Schneider, Ehlers, Seitz etc... ('80s, '90s)

- Two consistent threads:
  - Lens equation:
    - at least one image is made **brighter**
  - Optical scalar equations (Sachs 1961):
    - -> *focusing theorem* (Seitz et al. 1994)
    - Things viewed through 'clumpiness' are further than they appear...

Seitz, Schneider & Ehlers (1994)

Finally, we have derived an equation for the size of a light beam in a clumpy universe, relative to the size of a beam which is unaffected by the matter inhomogeneities. If we require that this second-order differential equation contains only the contribution by matter clumps as source term, the independent variable is uniquely defined and agrees with the χ-function previously introduced [see SEF, eq. (4.68)] for other reasons. This relative focusing equation immediately yields the result that a light beam cannot be less focused than a reference beam which is unaffected by matter inhomogeneities, prior to the propagation through its first conjugate point. In other words, no source can appear fainter to the observer than in the case that there are no matter inhomogeneities close to the line-of-sight to this source, a result previously demonstrated for the case of one (Schneider 1984) and several (Paper I, Seitz & Schneider 1994) lens planes.

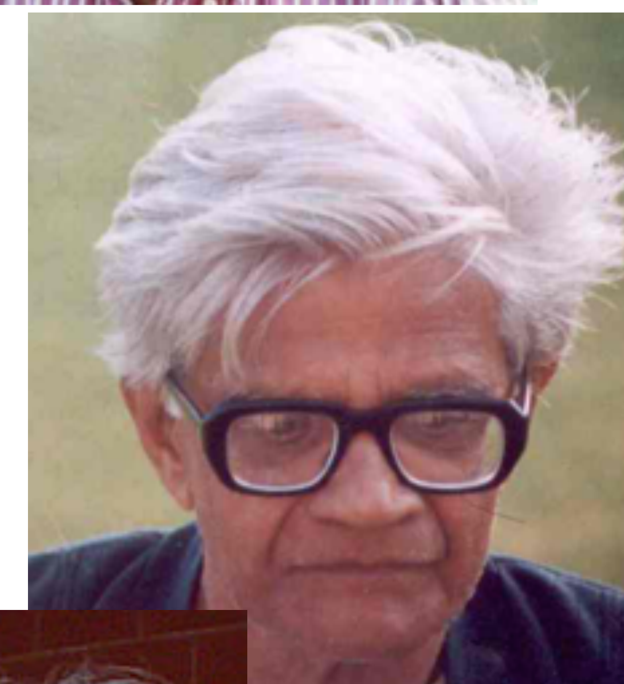
1992). Taking a somewhat different approach, Seitz, Schneider & Ehlers (1994) have used the optical scalars formalism of Sachs (1961) to show that the square root of the proper area of a narrow bundle of rays $D = \sqrt{A}$ obeys the 'focusing equation':

$$\boxed{\ddot{D}/D = -(R + \Sigma^2).} \tag{1}$$

Here $\ddot{D}$ is the second derivative of D with respect to affine distance along the bundle; $R = R_{\alpha\beta}k^\alpha k^\beta/2$ is the local Ricci focusing from matter in the beam, which for non-relativistic velocities is just proportional to the matter density; and Σ^2 is the squared rate of shear from the integrated effect of up-beam Weyl focusing – i.e. the tidal field of matter outside the beam. The resulting *focusing theorem* is that the RHS of (1) is non-positive, so that beams are always focused to smaller sizes, at least as compared to empty space-time,

More on the focusing theorem: $\ddot{D}/D = -(R + \Sigma^2)$

- Derived from Sachs '61 "optical scalars"
- from A.K. Raychaudhuri's (Landau) equation
  - transport of expansion, vorticity and shear
- $R = R_{ab}k^a k^b$ where R_{ab} is the *Ricci curvature*
  - local focusing by matter in the beam
- Σ^2 is the cumulative effect of *Weyl curvature*
  - i.e. the tidal effect of matter *outside* the beam
  - Σ being the *rate* of image shearing
- Like cosmological acceleration equation:
  - $d^2a/dt^2 = - 4\pi G(\varrho+3P/c^2)a$
  - so Σ^2 here plays the role of pressure???
- Also like Hawking-Ellis singularity theorem
  - both terms are positive => focusing
- e.g. Narlikar (Introduction to Relativity):
  - "*Thus the normal tendency of matter*
  - *is to focus light rays*"



Narlikar on the focusing theorem

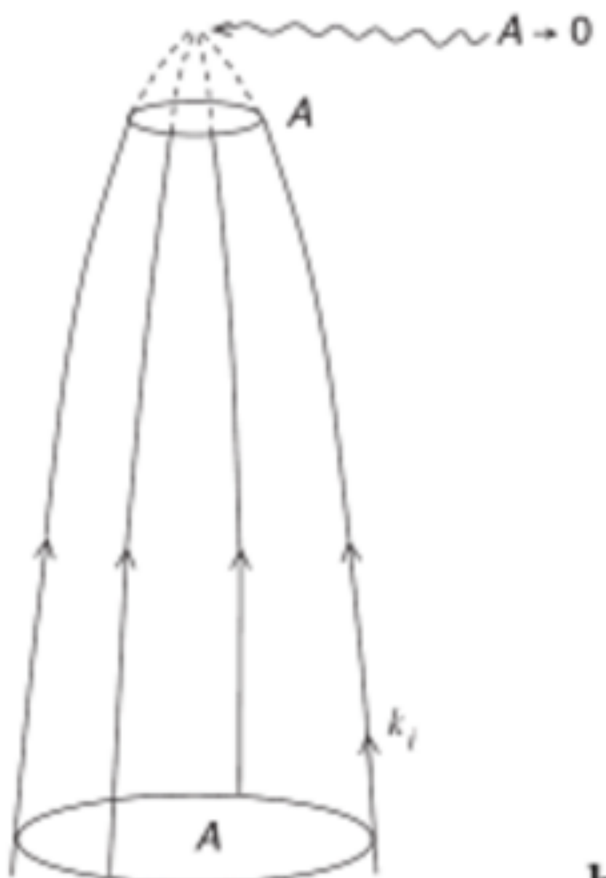


Fig. 18.3. The bundle of geodesics focusses in the future with its cross section A decreasing to zero. This effect was discussed in the context o spacetime singularity by A. K. Raychaudhuri.

The Raychaudhuri equation can be stated in a slightly different form as a *focussing theorem*. In this form it describes the effect of gravity on a bundle of null geodesics spanning a finite cross section. Denoting the cross section by A, we write the equation of the surface spanning the geodesics as $f =$ constant. Define the normal to the cross-sectional surface by $k_i = \partial f/\partial x^i$. Figure 18.3 shows the geometry of the bundle.

Using a calculation similar to that which led to the geodetic deviation equation in Chapter 5, we get the focussing equation as

$$\frac{1}{\sqrt{A}}\frac{\mathrm{d}^2\sqrt{A}}{\mathrm{d}\lambda^2} = \frac{1}{2}R_{im}k^i k^m - |\sigma|^2, \tag{18.10}$$

Equation (18.10) is similar to the Raychaudhuri equation with $|\sigma|^2$ being the square of the magnitude of shear. With Einstein's equations, we can rewrite (18.10) as

$$\frac{1}{\sqrt{A}}\frac{\mathrm{d}^2\sqrt{A}}{\mathrm{d}\lambda^2} = -4\pi G\left(T_{im} - \frac{1}{2}g_{im}T\right)k^i k^m - |\sigma|^2. \tag{18.12}$$

For dust we have $T_{im} = \rho u_i u_m$ and this condition is satisfied with the left-hand side equalling $\rho(u_i k^i)^2$. (Remember that k_i is a null vector, so $g_{im}k^i k^m = 0$.) Thus the normal tendency of matter is to focus light rays by gravity.

even more on the focusing theorem: $\ddot{D}/D = -(R + \Sigma^2)$

- Schneider et al are <u>adding</u> lenses to a background - no compensation.
  - e.g. discussion is Schneider, Ehlers & Falco book "the apparent contradiction ... disappears .... masses .... change the geometry.."
- Does this explain the apparent conflict with flux conservation?
- No. In a cosmological context we are interested in how D deviates from the background value $D = D_0 + D_1 + \ldots$
- If we take the average, and linearise,
  - and assuming $\langle \delta R \rangle = 0$ we have the *averaged focusing theorem*

$$\langle \ddot{D} \rangle / D_0 = -\langle \Sigma^2 \rangle < 0.$$

- There is an inevitable tendency for beams to focus
- Not difficult to show that this predicts decrease of distance that is *qualitatively* the same as found by Clarkson et al. 2014
  - i.e. a big - and possibly divergent - effect!
- So Weinberg was wrong?

GRAVITATIONAL MAGNIFICATION OF THE COSMIC MICROWAVE BACKGROUND

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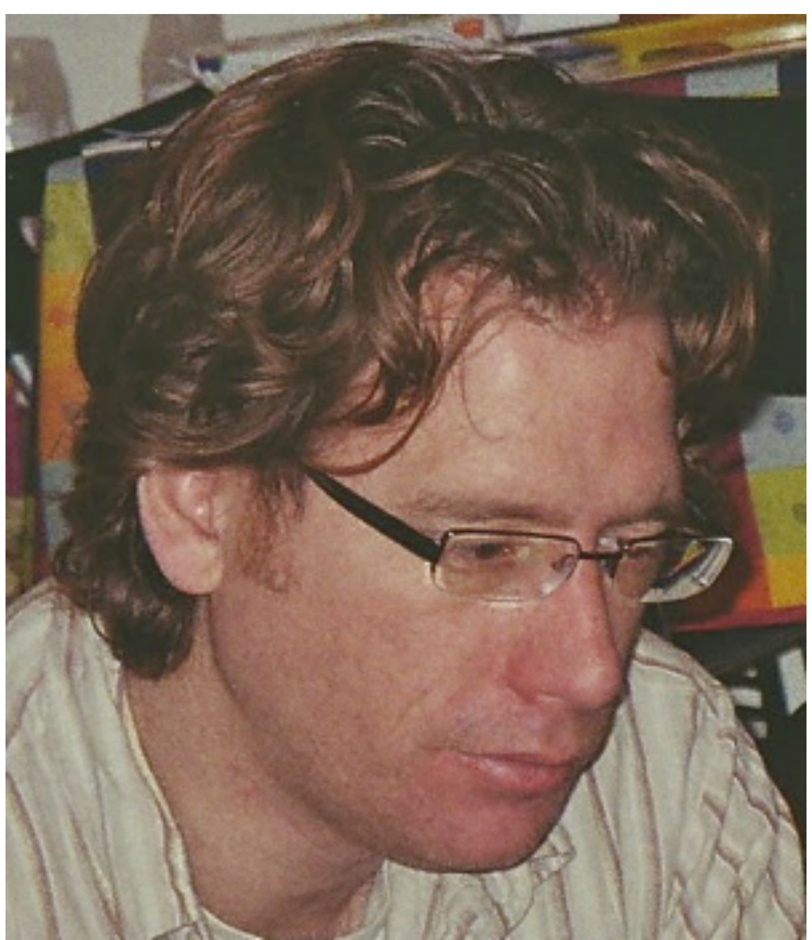
ABSTRACT

Some aspects of gravitational lensing by large-scale structure are investigated. We show that lensing causes the damping tail of the cosmic microwave background (CMB) power spectrum to fall less rapidly with decreasing angular scale than previously expected. This is because of a transfer of power from larger to smaller angular scales, which produces a fractional change in power spectrum that increases rapidly beyond $\ell \sim 2000$. We also find that lensing produces a nonzero mean magnification of structures on surfaces of constant redshift if weighted by area on the sky. This is a result of the fact that light rays that are evenly distributed on the sky oversample overdense regions. However, this mean magnification has a negligible affect on the CMB power spectrum. A new expression for the lensed power spectrum is derived, and it is found that future precision observations of the high-ℓ tail of the power spectrum will need to take lensing into account when determining cosmological parameters.

Subject headings: cosmic microwave background — gravitational lensing



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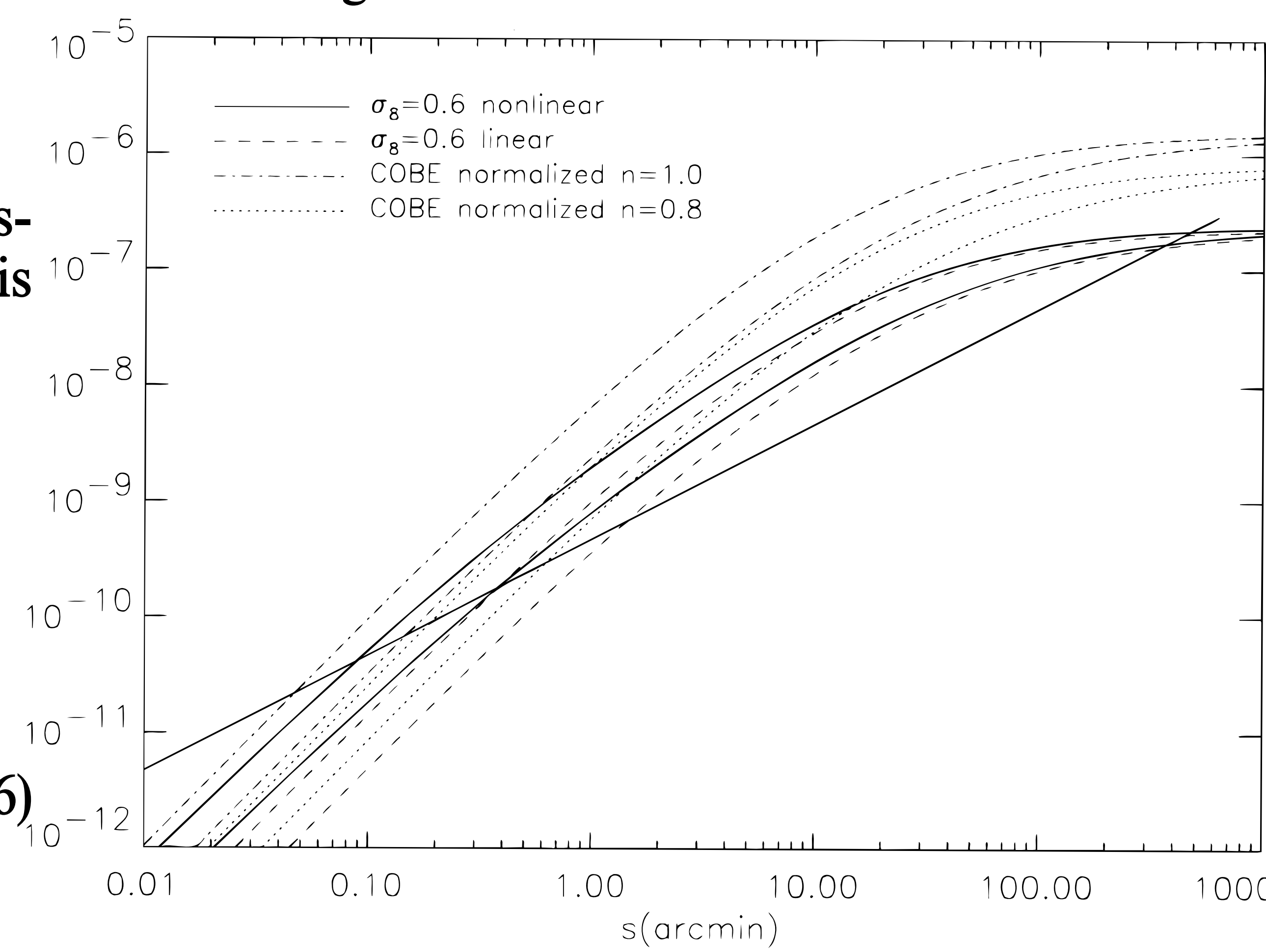
ABSTRACT

Some aspects of gravitational lensing by large-scale structure are investigated. We show that lensing causes the damping tail of the cosmic microwave background (CMB) power spectrum to fall less rapidly with decreasing angular scale than previously expected. This is because of a transfer of power from larger to smaller angular scales, which produces a fractional change in power spectrum that increases rapidly beyond $\ell \sim 2000$. We also find that lensing produces a nonzero mean magnification of structures on surfaces of constant redshift if weighted by area on the sky. This is a result of the fact that light rays that are evenly distributed on the sky oversample overdense regions. However, this mean magnification has a negligible affect on the CMB power spectrum. A new expression for the lensed power spectrum is derived, and it is found that future precision observations of the high-ℓ tail of the power spectrum will need to take lensing into account when determining cosmological parameters.

Subject headings: cosmic microwave background — gravitational lensing

unperturbed path. The shear tensor that measures the distortion and expansion of an infinitesimally thin beam is then

$$\Phi_{ij} \equiv \frac{\partial \delta\theta_i}{\partial \theta_j} = \Phi^o_{ij} + \Delta\Phi_{ij} = \frac{-2}{g(r)} \int_0^r dr' g(r') g(r - r') \phi_{,ij}(r')$$
$$+ \frac{4}{g(r)} \int_0^r dr' \int_0^{r'} dr'' g(r - r') g(r' - r'')$$
$$\times [g(r')\phi_{,k}(r'')\phi_{,ijk}(r') + g(r'')\phi_{,jk}(r'')\phi_{,ik}(r')] \,. \qquad (6)$$



Kibble & Lieu (2005)

AVERAGE MAGNIFICATION EFFECT OF CLUMPING OF MATTER

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ABSTRACT

The aim of this paper is to reexamine the question of the average magnification in a universe with some inhomogeneously distributed matter. We present an analytic proof, valid under rather general conditions, including clumps of any shape and size and strong lensing, that as long as the clumps are uncorrelated, the average "reciprocal" magnification (in one of several possible senses) is precisely the same as in a homogeneous universe with an equal mean density. From this result, we also show that a similar statement can be made about one definition of the average "direct" magnification. We discuss, in the context of observations of discrete and extended sources, the physical significance of the various different measures of magnification and the circumstances in which they are appropriate.

Subject headings: cosmology: miscellaneous — distance scale — galaxies: distances and redshifts — gravitational lensing

Kibble & Lieu 2005

There is another important distinction to be made. We may choose at random one of the sources at redshift z, or we may choose a random direction in the sky and look for sources there. These are not the same; the choices are differently weighted. If one part of the sky is more magnified, or at a closer angular-size distance, the corresponding area of the constant-z surface will be smaller, so fewer sources are likely to be found there. In other words, choosing a source at random will give on average a smaller magnification or larger angular-size distance.

- Weinberg: $\langle\mu\rangle = 1$ when averaged over *sources* (or area)
- Kibble & Lieu: $\langle 1/\mu\rangle = 1$ when averaged over *directions on the sky*
  - latter is more relevant for CMB observations
  - but strictly only valid in weak lensing regime

Recap of historical review

- Zel'dovich '63 .... Feynman & Gunn .... Kantowski ... Dyer & Roeder
  - structure makes things look *fainter* on average
  - for opaque lenses at least
- Weinberg '76 - *no effect* for transparent lenses (flux conservation)
- Schneider et al. ('84..'94) (from Raychaudhuri, Sachs, Narlikar):
  - *magnification* and *focusing theorems*
  - structure makes things look *nearer* (i.e. *brighter*)- a <u>big</u> effect
- Ellis, Bassett & Dunsby '97
  - critique of Weinberg '76 - effect of strong lensing on small scales
- Metcalf and Silk '97: mean magnification of the CMB = $0 + O(\theta^2) \sim 10^{-6}$
- Kibble & Lieu '05
  - distinction between *source* and *direction* averages
    - Weinberg: $\langle\mu\rangle = 1$ averaged over *sources* (or area on source sphere)
    - K+L: $\langle 1/\mu \rangle = 1$ when averaged over *directions* (as e.g. for CMB)
- Outstanding questions:
  - How do we make sense of these apparently conflicting results?
  - What is the relation to recent results from 2nd order Pert^n Theory?

Recent developments...

- Relativists: "have cosmologists erred in failing to take into account the inherent non-linearity of Einstein's equations?"
  - cosmologists tend to do background + linear theory calculations
  - but Einstein's equations (metric <-> matter) are non-linear
  - averaging and non-linearity "do not commute"
  - so maybe *dark energy* is a mirage?
- requires calculations in 2nd order perturbation theory (v. technical)
- now mostly accepted that effects are too small to explain acceleration
- but maybe there are still interesting percent level effects:
  - Clarkson, Ellis++ '12 - large ($O(\kappa^2)$) source magnification
  - Clarkson++ '14 - similarly large z-surface *area* increase
    - violates Weinberg's assumption
  - "backreaction" strikes back?
- and the size of the effect is qualitatively consistent with expectation of the *focusing theorem* (Schneider, Ehlers & Sietz)

$$
\begin{aligned}
\hat{D}_A = {} & a(\chi_s)\chi_s\Big\{1+\left(1-\frac{1}{\mathcal{H}_s\chi_s}\right)\Phi_o+\frac{1}{2}\Big[\left(1-\frac{1}{\mathcal{H}_s\chi_s}\right)\Phi_o^{(2)}-\left(1-\frac{1}{\mathcal{H}_s\chi_s}\right)\omega_{\|o}-\left(\frac{\mathcal{H}'}{\mathcal{H}^2}-7\right)\Phi_o^2\Big]-\left(2-\frac{1}{\mathcal{H}_s\chi_s}\right)\Phi_s \\
& +\frac{1}{2}\Big[-\Psi_s^{(2)}-\left(1-\frac{1}{\mathcal{H}_s\chi_s}\right)\Phi_s+\frac{1}{2}\left(1-\frac{2}{\mathcal{H}_s\chi_s}\right)\omega_{\|s}-\frac{1}{2}h_{\|s}-\left(\frac{\mathcal{H}'}{\mathcal{H}^2}-\frac{2}{\chi_s\mathcal{H}_s}-7\right)\Phi_s^2-2\left(1-\frac{2}{\chi_s\mathcal{H}_s}\right)\chi_s\Phi_s\Phi_s' \\
& -2\chi_s\Phi_s\nabla_\|\Phi_s\Big]+\frac{1}{\mathcal{H}_s\chi_s}\nabla_\| v_o+\frac{1}{2}\Big[\frac{1}{\mathcal{H}_s\chi_s}\nabla_\| v_o^{(2)}-\frac{1}{\mathcal{H}_s\chi_s}v_{\|o}-\left(\frac{\mathcal{H}'}{\mathcal{H}^2}-2\right)\nabla_\| v_o\nabla_\| v_o+\frac{1}{\chi_s\mathcal{H}_s}\nabla_{\perp i}v_o\nabla_\perp^i v_o\Big] \\
& +\left(1-\frac{1}{\mathcal{H}_s\chi_s}\right)\nabla_\| v_s+\frac{1}{2}\Big[\left(1-\frac{1}{\mathcal{H}_s\chi_s}\right)\nabla_\| v_s^{(2)}+\left(1-\frac{1}{\mathcal{H}_s\chi_s}\right)v_{\|s}+\left(\frac{\mathcal{H}'}{\mathcal{H}^2}-\frac{2}{\chi_s\mathcal{H}_s}-1\right)\nabla_\| v_s\nabla_\| v_s \\
& +2\chi\left(1-\frac{1}{\chi_s\mathcal{H}_s}\right)\nabla_\| v_s\left(\nabla_\| v_s'-\nabla_\|^2 v_s\right)+\left(1-\frac{1}{\chi_s\mathcal{H}_s}\right)\nabla_{\perp i}v_s\nabla_\perp^i v_s-2\left(\frac{\mathcal{H}'}{\mathcal{H}^2}-\frac{1}{\chi_s\mathcal{H}_s}+4\right)\Phi_s\Phi_o \\
& +2\left(1-\frac{2}{\chi_s\mathcal{H}_s}\right)\chi_s\Phi_o\Phi_s'-2\chi\Phi_o\nabla_\|\Phi_s-2\left(\frac{\mathcal{H}'}{\mathcal{H}^2}-\frac{1}{\chi_s\mathcal{H}_s}\right)\nabla_\| v_s\nabla_\| v_o-2\chi_s\left(1-\frac{1}{\chi_s\mathcal{H}_s}\right)\nabla_\| v_o\left(\nabla_\| v_s'-\nabla_\|^2 v_s\right) \\
& -2\left(\frac{\mathcal{H}'}{\mathcal{H}^2}+\frac{1}{2\chi_s\mathcal{H}_s}+\frac{11}{2}\right)\Phi_o\nabla_\| v_o-2\left(\frac{\mathcal{H}'}{\mathcal{H}^2}-\frac{1}{\chi_s\mathcal{H}_s}+3\right)\Phi_s\nabla_\| v_s+2\left(\frac{\mathcal{H}'}{\mathcal{H}^2}-1\right)\Phi_o\nabla_\| v_s+2\chi_s\left(1-\frac{2}{\chi_s\mathcal{H}_s}\right)\Phi_s'\nabla_\| v_s \\
& +2\chi_s\nabla_\|\Phi_s\nabla_\| v_s-2\chi_s\left(1-\frac{1}{\chi_s\mathcal{H}_s}\right)\Phi_s(\nabla_\| v_s'-\nabla_\|^2 v_s)-2\chi_s\left(1-\frac{2}{\chi_s\mathcal{H}_s}\right)\Phi_s'\nabla_\| v_o-2\chi_s\nabla_\|\Phi_s\nabla_\| v_o \\
& +2\left(\frac{\mathcal{H}'}{\mathcal{H}^2}+2\right)\Phi_s\nabla_\| v_o+2\left(1-\frac{1}{\chi_s\mathcal{H}_s}\right)\chi_s\Phi_o(\nabla_\| v_s'-\nabla_\|^2 v_s)\Big]+\frac{2}{\chi_s}\int_\chi\Phi+\frac{1}{2}\Big[\frac{1}{\chi_s}\int_\chi(\Phi^{(2)}+\Psi^{(2)}) \\
& -\frac{1}{\chi_s}\int_\chi\frac{(\chi-\chi_s)}{\chi}\omega_\|+\frac{1}{\chi_s}\int_\chi 3\frac{(\chi-\chi_s)}{\chi}h_\|\Big]-2\left(1-\frac{1}{\mathcal{H}_s\chi}\right)\int_\chi\Phi'+\frac{1}{2}\Big[-\left(1-\frac{1}{\mathcal{H}_s\chi_s}\right)\left(\int_\chi(\Phi'_{(2)}+\Psi'_{(2)})\right) \\
& -\int_\chi\left(1-\frac{1}{\mathcal{H}_s\chi_s}+\frac{(2\chi-\chi_s)}{2\chi_s}\right)\omega_\|^{(2)\prime}+\int_\chi\left(1-\frac{1}{\mathcal{H}_s\chi_s}-\frac{(2\chi-\chi_s)}{\chi_s}\right){h_\|}'\Big]+\int_\chi\frac{(\chi-\chi_s)\chi}{\chi_s}\nabla_\perp^2\Phi \\
& +\frac{1}{2}\Big[\frac{1}{2}\int_\chi\frac{(\chi-\chi_s)\chi}{\chi_s}\nabla_\perp^2(\Phi^{(2)}+\Psi^{(2)})-\frac{1}{2}\int_\chi\frac{(\chi-\chi_s)\chi}{\chi_s}\nabla_\perp^2\omega_\|-\frac{1}{2}\int_\chi\frac{(\chi-\chi_s)\chi}{\chi_s}\nabla_\perp^2 h_\|\Big]+\frac{1}{2}\Big\{-2\Phi_o\Big[\left(1+\frac{2}{\chi_s\mathcal{H}_s}\right)\frac{2}{\chi}\int_\chi\Phi \\
& +2\left(\frac{\mathcal{H}'}{\mathcal{H}^2}+\frac{3}{\chi_s\mathcal{H}_s}-4\right)\int_\chi\Phi'-\left(2+\frac{1}{\chi_s\mathcal{H}_s}\right)\int_\chi\chi\nabla_\perp^2\Phi+\left(9+\frac{2}{\chi_s\mathcal{H}_s}\right)\int_\chi\frac{(\chi-\chi_s)\chi}{\chi_s}\nabla_\perp^2\Phi+4\int_\chi\frac{\chi}{\chi_s}\Phi' \\
& +4\int_\chi\frac{(\chi-\chi_s)}{\chi_s}\Phi'-4\int_\chi\frac{(\chi-\chi_s)\chi}{\chi_s}\Phi''\Big]+2\Phi_s\Big[\left(1-\frac{1}{\chi_s\mathcal{H}_s}\right)\frac{4}{\chi}\int_\chi\Phi+2\left(\frac{\mathcal{H}'}{\mathcal{H}^2}-\frac{2}{\chi_s\mathcal{H}_s}+2\right)\int_\chi\Phi' \\
& -\left(2+\frac{1}{\chi_s\mathcal{H}_s}\right)\int_\chi\chi\nabla_\perp^2\Phi+2\left(1-\frac{1}{\chi_s\mathcal{H}_s}\right)\int_\chi\frac{(\chi-\chi_s)\chi}{\chi_s}\nabla_\perp^2\Phi\Big]+2\nabla_\| v_o\Big[\frac{2}{\chi}\left(1-\frac{2}{\chi_s\mathcal{H}_s}\right)\int_\chi\Phi-2\int_\chi\Phi' \\
& -\left(2-\frac{1}{\chi_s\mathcal{H}_s}\right)\int_\chi\chi\nabla_\perp^2\Phi+2\left(2-\frac{1}{\chi_s\mathcal{H}_s}\right)\int_\chi\frac{(\chi-\chi_s)\chi}{\chi_s}\nabla_\perp^2\Phi\Big]-2\nabla_\| v_s\Big[\frac{2}{\chi_s}\left(5-\frac{1}{\chi_s\mathcal{H}_s}\right)\int_\chi\Phi-4\left(1-\frac{1}{\chi_s\mathcal{H}_s}\right)\int_{\lambda_s}\Phi' \\
& -\left(2-\frac{1}{\chi_s\mathcal{H}_s}\right)\int_\chi\chi\nabla_\perp^2\Phi+\left(3-\frac{2}{\chi_s\mathcal{H}_s}\right)\int_\chi\frac{(\chi-\chi_s)\chi}{\chi_s}\nabla_\perp^2\Phi\Big]+4\left(1-\frac{1}{\chi_s\mathcal{H}_s}\right)\nabla_\perp v_s\int_\chi\nabla_\perp^i\Phi \\
& +4\Big[\frac{2}{\chi_s}\left(3-\frac{2}{\chi_s\mathcal{H}_s}\right)\int_\chi\Phi+\left(\frac{\mathcal{H}'}{\mathcal{H}^2}+\frac{1}{\chi_s\mathcal{H}_s}-2\right)\int_\chi\Phi'-\left(2-\frac{1}{\chi_s\mathcal{H}_s}\right)\int_\chi\chi\nabla_\perp^2\Phi+\left(3-\frac{2}{\chi_s\mathcal{H}_s}\right)\int_\chi\frac{(\chi-\chi_s)\chi}{\chi_s}\nabla_\perp^2\Phi\Big]\int_\chi\Phi' \\
& -4\chi\left(1-\frac{2}{\chi_s\mathcal{H}_s}\right)\chi_s\Phi_s'\int_\chi\Phi'-4\chi_s\nabla_\|\Phi_s\int_\chi\Phi'-4\left(1-\frac{1}{\chi_s\mathcal{H}_s}\right)\left[\chi\left(\nabla_\| v_s'-\nabla_\|^2 v_s\right)\right]\int_\chi\Phi'+16\int_\chi\frac{(\chi-\chi_s)}{\chi_s}\Phi\Phi' \\
& +\frac{2}{\chi_s}\int_\lambda\Phi^2-4\int_\chi\frac{1}{\chi\chi_s}\Phi\int_{\tilde\chi}\Phi(\tilde\chi)+8\int_\chi\Phi'\int_{\tilde\chi}\Phi'(\tilde\chi)+8\int_\chi\Phi\Phi'-4\int_\chi\frac{(\chi-\chi_s)}{\chi\chi_s}\left[\frac{1}{\chi}\Phi\int_{\tilde\chi}\Phi(\tilde\chi)+\Phi^2-2\Phi'\int_{\tilde\chi}\Phi(\tilde\chi)\right] \\
& -4\int_\chi\frac{1}{\chi}\Phi\int_{\tilde\chi}\frac{(\tilde\chi-\chi)\tilde\chi}{\chi_s}\nabla_\perp^2\Phi(\tilde\chi)-4\int_\chi\frac{(\chi-\chi_s)}{\chi^2\chi_s}\Phi\int_{\tilde\chi}(\tilde\chi-\chi)\tilde\chi\nabla_\perp^2\Phi(\tilde\chi)+4\int_\chi\frac{(\chi-\chi_s)^2}{\chi_s}\Phi\nabla_\perp^2\Phi \\
& -4\int_\chi\Phi\int_{\tilde\chi}\frac{\tilde\chi}{\chi}\nabla_\perp^2\Phi(\tilde\chi)+14\int_\chi\frac{(\chi-\chi_s)\chi}{\chi_s}\Phi\nabla_\perp^2\Phi+2\int_\chi\frac{(\chi-\chi_s)}{\chi_s}\Phi\int_{\tilde\chi}\tilde\chi\nabla_\perp^2\Phi(\tilde\chi)+4\int_\chi\frac{(\chi-\chi_s)}{\chi_s}\nabla_\perp^2\Phi\int_{\tilde\chi}\Phi(\tilde\chi) \\
& +4\int_\chi\frac{(\chi-\chi_s)}{\chi_s}\Phi'\int_{\tilde\chi}\tilde\chi\nabla_\perp^2\Phi(\tilde\chi)+4\int_\chi\frac{(\chi-\chi_s)}{\chi}\Phi\int_{\tilde\chi}\frac{(\tilde\chi-\chi)\tilde\chi}{\chi}\nabla_\perp^2\Phi(\tilde\chi)-8\int_\chi\frac{(\chi-\chi_s)}{\chi_s}\nabla_{\perp i}\Phi\int_{\tilde\chi}\frac{\tilde\chi}{\chi}\nabla_\perp^i\Phi(\tilde\chi) \\
& -8\int_\chi\frac{\chi}{\chi_s}\nabla_{\perp i}\Phi\int_{\tilde\chi}\nabla_\perp^i\Phi(\tilde\chi)-\frac{8}{\chi_s}\int_\chi\int_{\tilde\chi}\nabla_{\perp i}\Phi(\tilde\chi)\int_{\tilde\chi}\nabla_\perp^i\Phi(\tilde\chi)\,,+12\int_\chi\frac{(\chi-\chi_s)\chi}{\chi_s}\nabla_{\perp i}\Phi\nabla_\perp^i\Phi \\
& -8\int_\chi\frac{(\chi-\chi_s)\chi}{\chi_s}\nabla_{\perp i}\Phi'\int_{\tilde\chi}\nabla_\perp^i\Phi(\tilde\chi)-8\left(1-\frac{1}{\chi_s\mathcal{H}_s}\right)\int_{\tilde\chi}\frac{\tilde\chi}{\chi}\nabla_\perp^i\Phi'(\tilde\chi)\int_{\tilde\chi}\nabla_\perp^i\Phi(\tilde\chi) \\
& -4\int_\chi\frac{(\chi-\chi_s)\chi}{\chi_s}\int_{\tilde\chi}\nabla_{\perp\langle i}\nabla_{\perp j\rangle}\Phi(\tilde\chi)\int_{\tilde\chi}\nabla_\perp^{\langle i}\nabla_\perp^{j\rangle}\Phi(\tilde\chi)-2\int_\chi\frac{(\chi-\chi_s)}{\chi_s}\nabla_\perp^2\Phi\int_{\tilde\chi}(\tilde\chi-\chi)\tilde\chi\nabla_\perp^2\Phi(\tilde\chi) \\
& +8\int_\chi\frac{(\chi-\chi_s)}{\chi_s}\int_{\tilde\chi}\nabla_{\perp i}\Phi(\tilde\chi)\int_{\tilde\chi}\frac{\tilde\chi^2}{\chi}\nabla_\perp^i\nabla^2\Phi(\tilde\chi)-4\int_\chi\frac{(\chi-\chi_s)}{\chi_s}\nabla_{\perp i}\Phi\int_{\tilde\chi}\frac{(\tilde\chi-\chi)\tilde\chi^2}{\chi}\nabla_\perp^i\nabla_\perp^2\Phi(\tilde\chi)\Big\}\Big\}\,.
\end{aligned}
\tag{B1}
$$

What is the distance to the CMB?
How relativistic corrections remove the tension with local H_0 measurements

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The success of precision cosmology depends not only on accurate observations, but also on the theoretical model – which must be understood to at least the same level of precision. Subtle relativistic effects can lead to biased measurements if they are neglected. One such effect gives a systematic shift in the distance-redshift relation away from its background value, due to the accumulation of all possible lensing events. We estimate the expectation value of this aggregated lensing using second-order perturbations about a concordance background, and show that the distance to last scattering is shifted by several percent. Neglecting this shift leads to significant bias in the background cosmological parameters. We show that this removes the tension between local measurements of H_0 and those measured through the CMB and favours a closed universe.

Clarkson et al. 2014

$$\langle \Delta \rangle \simeq \frac{3}{2} \left\langle \left(\frac{\delta d_A}{\chi_s} \right)^2 \right\rangle = \frac{3}{2} \left\langle \kappa^2 \right\rangle \,, \tag{1.5}$$

where κ is the usual linear lensing convergence. This is actually the leading contribution to the expected change to large distances. We prove this remarkably simple and important result in a variety of ways in several appendices. It implies that the total area of a sphere of constant redshift will be larger than in the background. Physically this is because a sphere about us in redshift space is not a sphere in real space — lensing implies that this 'sphere' becomes significantly crumpled in real space, and hence has a larger area. When interpreted

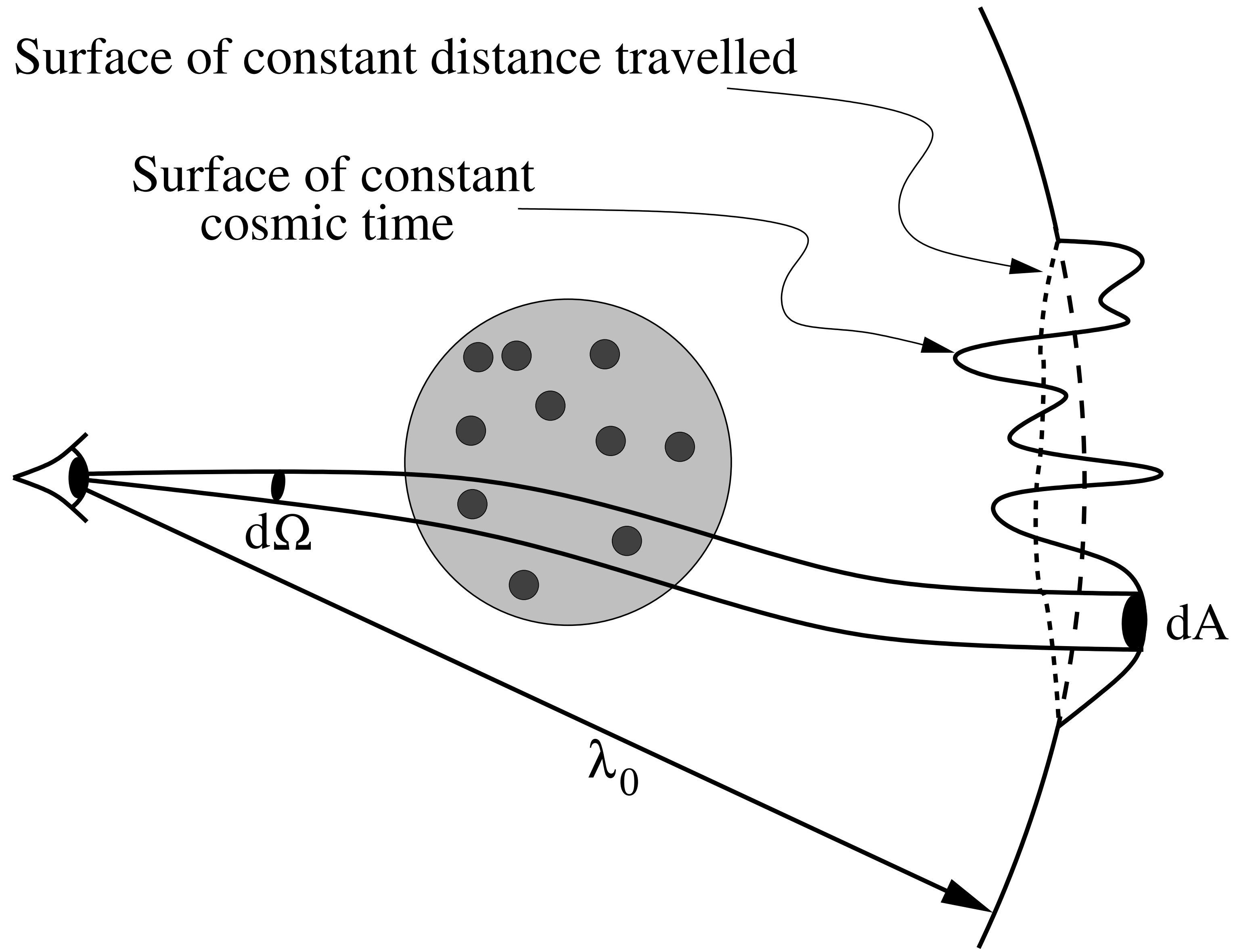
4 Conclusions

We have demonstrated an important overall shift in the distance redshift relation when the aggregate of all lensing events is considered, calculated by averaging over an ensemble of universes. This result is a consequence of flux conservation at second-order in perturbation theory. This is a purely relativistic effect with no Newtonian counterpart — and it is the first quantitative prediction for a significant change to the background cosmology when averaging over structure [21]. The extraordinary amplification of aggregated lensing comes mainly from the integrated lensing of structure on scales in the range 1–100 Mpc, although structure down to 10kpc scales contributes significantly. We have estimated the size of the effect using

NK + Peacock 2015

- Weinberg *assumes* that the area of a surface of constant redshift is unperturbed by lensing by intervening structures
    - same assumption is made by Kibble & Lieu
    - seems reasonable since *static* lenses do not affect redshift
    - and leads to conservation of e.g. source-averaged flux density
        - but not strictly true and breaks down at some level
- What *is* the change in the area of the constant-z surface (or cosmic photosphere) caused by structure?

KP2015: closing the loophole in Weinberg's argument



2 effects:

1) wiggly lines don't get as far as straight lines
2) wrinkly surface has more area than a smooth one

but both effects are ~(bending angle)^2 ~ 10^{-6}

What is the area of a wavy surface?

Key features of KP15 calculation of area of photosphere

- Calculations are rather technical, some key features are:
  - Weak field assumption:
    - we model the metric as weak field limit of GR
      - but we allow for non-rel motion of sources
        - these have negligible effects
      - similarly for gravitational waves
        - "photons can't surf a gravitational wave"
      - going beyond 1st order can be estimated and is tiny effect
    - *the problem is isomorphic to light propagation in "lumpy glass"*
  - Boundary conditions:
    - Perturbation theory calculations assume photosphere is constant z
    - Not true. It is more realistically a surface of constant cosmic time
    - Pert. theo. results may be qualitatively OK, but fail quantitatively
  - Final result for perturbation to the area of the photosphere is

$$\langle \Delta A \rangle / A_0 = \frac{1}{\lambda_0^2} \int\limits_0^{\lambda_0} d\lambda \, (2\lambda(\lambda_0 - \lambda) + \lambda^2) J(\lambda).$$

where

$$J \equiv -8 \int\limits_{-\infty}^{0} dy \, \xi'_\phi(y)/y = 2\pi \int k \, \Delta_\phi^2(k) \, d\ln k,$$

but J = d<θ^2>/dλ and Jλ is on the order of 10^{-6}

NK + Peacock 2015 - 2nd point

- Perturbation to the *area* is on the order of the mean squared cumulative deflection angle
- This is a one-part-in-a-million effect
  - and dominated by large-scale structure
- Relativistic perturbation theory, *focussing theorem* etc. give perturbation to the distance that is on the order of the mean squared shear (or convergence)
  - this is much larger
  - and dominated by small-scale structure (possibly divergent)
- All claims for large effects are *purely statistical effects*:
  - The mean flux magnification μ of a source is unity
    - so $\langle \Delta\mu \rangle_{\text{source}} = 0$
  - but μ is a fluctuating quantity
  - so any non-linear function of μ (e.g. $D/D_0 = 1 / \sqrt{\mu}$) will *not* average to unity

KP15: Statistical biases...

- Example: Source averaged distance bias:
  - $D/D_0 = \mu^{-1/2} = (1 + \Delta\mu)^{-1/2} = 1 - \Delta\mu / 2 + 3(\Delta\mu)^2/8 + \ldots$
  - so $\langle D/D_0\rangle_{source} = 1 + 3\langle(\Delta\mu)^2\rangle/8 + \ldots = 1 + 3\langle\varkappa^2\rangle/2 + \ldots$
- Similarly for source averaged mean inverse magnification
  - $\langle D^2/D_0^2\rangle_{source} = 1 + 4 \langle\varkappa^2\rangle + \ldots$
- *These are precisely the results for the mean perturbation to the distance and distance squared found by Clarkson et al. 2014*
- But e.g. the latter is not the perturbation to the constant z surface area
  - that would be the average over *directions* rather than over sources
- Similarly, Clarkson et al. 2012 claim mean source averaged flux magnification is $\langle\mu\rangle = 1 + \langle 3\varkappa^2 + \gamma^2\rangle + \ldots = 1 + \langle 4\varkappa^2\rangle + \ldots.$
  - but this is the *direction* averaged magnification
- *These come from non-commutativity of averaging and non-linearity*
  - $\langle f(x)\rangle$!= $f(\langle x\rangle)$ if x is a fluctuating quantity
  - and have nothing to do with the non-linearity of Einstein's equations

What about the "*focusing theorem*"? $\langle \ddot{D} \rangle / D_0 = -\langle \Sigma^2 \rangle < 0.$

- 2 lessons from foregoing:
  - 1) The theorem applies to a bundle of rays fired along a given direction
    - i.e. a *direction* - not *source*-averaged quantity
    - and paths to sources avoid over-densities
    - so care is needed in interpreting this
  - 2) D is a non-linear function of A
    - so, because A is a fluctuation quantity, we automatically expect a statistical bias in D
    - and the size of the effect is ~ $\langle \varkappa^2 \rangle$
- So is there a "normal tendency of matter to focus light rays"?
  - as inferred from the averaged focusing theorem
- or is this simply a statistical effect?

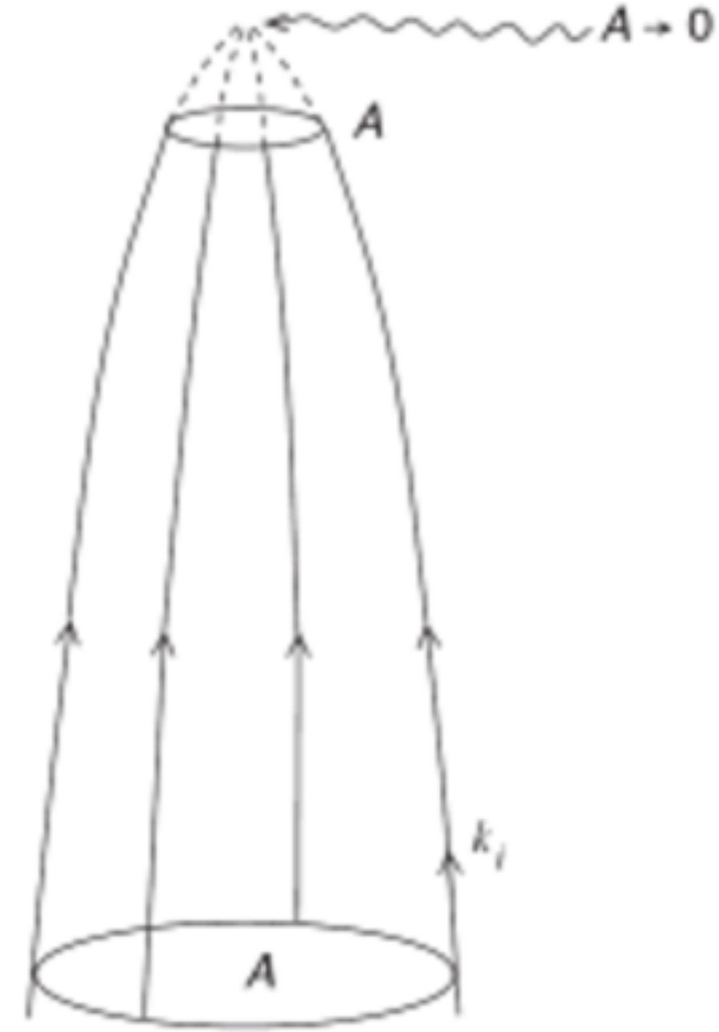


Fig. 18.3. The bundle of geodesics focusses in the future with its cross section *A* decreasing to zero. This effect was discussed in the context of spacetime singularity by A. K. Raychaudhuri.

KP15 on the "*focusing theorem*"? $\langle \ddot{D} \rangle / D_0 = -\langle \Sigma^2 \rangle < 0.$

- We have developed the optical scalar transport equations in a form appropriate when one wishes to specify the metric fluctuations as a stochastic random field (with zero mean for k=0 component)
    - interesting subtlety: one should *not* assume $\langle \delta R \rangle = 0$
    - in inflationary context, small scale space-time curvature fluctuations have to accommodate themselves within the (flat-space) boundary conditions imposed when the larger regions accelerate outside of horizon
- We have solved these to obtain the ensemble average of the perturbation to the *area* of a beam of specified solid angle fired off from the observer and propagating back to the source surface.
- We perform a double expansion, working to second order in δ(metric) and to lowest order in the inverse of "coherence scale"/Hubble scale
- Cancellation: Not just "Born level", but 1st "beyond Born" also
- We were only able to solve for the case where metric fluctuations are non-evolving (like in Einstein - de Sitter) but were able to obtain the "un-focusing theorem": $\langle \Delta A/A \rangle = -2J\lambda/3 + ...$
    - this is consistent with the more general result (variable J) found by more straightforward approach.
- An exactly analogous calculation for $\langle \Delta D/D \rangle$ does not show cancellation and results in much larger ($O(\varkappa^2)$) result. *But just the statistical bias*. QED

Optical scalars (in weak-field GR or lumpy glass)

$\ddot{\mathbf{r}} = \boldsymbol{\nabla}_\perp \tilde{n}$ Geodesic equation

$$n = [(1 - 2\phi(\mathbf{r})/c^2)/(1 + 2\phi(\mathbf{r})/c^2)]^{1/2}$$

Optical *tensor* transport equation:

$$\dot{\mathbf{K}} = (\boldsymbol{\nabla}_\mathbf{x}\boldsymbol{\nabla}_\mathbf{x} - \mathbf{K}\partial_z)\tilde{n} - \boldsymbol{\nabla}_\mathbf{x}\tilde{n}\boldsymbol{\nabla}_\mathbf{x}\tilde{n} - \mathbf{K}\cdot\mathbf{K}$$

Optical scalar transport equations:

$$\dot{\theta} = \left(\frac{\nabla_\perp{}^2}{2} - \theta\partial_\lambda\right)\tilde{n} - |\boldsymbol{\nabla}_\perp\tilde{n}|^2/2 - \theta^2 - \Sigma^2$$

$$\dot{\boldsymbol{\Sigma}} = (\{\boldsymbol{\nabla}_\perp\boldsymbol{\nabla}_\perp\} - \boldsymbol{\Sigma}\partial_\lambda)\tilde{n} - \{\boldsymbol{\nabla}_\perp\tilde{n}\boldsymbol{\nabla}_\perp\tilde{n}\} - 2\theta\boldsymbol{\Sigma}$$

Solve for θ

The solution of $\dot{A}/2A = \theta(\lambda) = \lambda^{-1} + \Delta\theta(\lambda)$ is

$$A = \Omega\lambda^2 \exp\left(2\int_0^\lambda d\lambda'\,\Delta\theta(\lambda')\right)$$

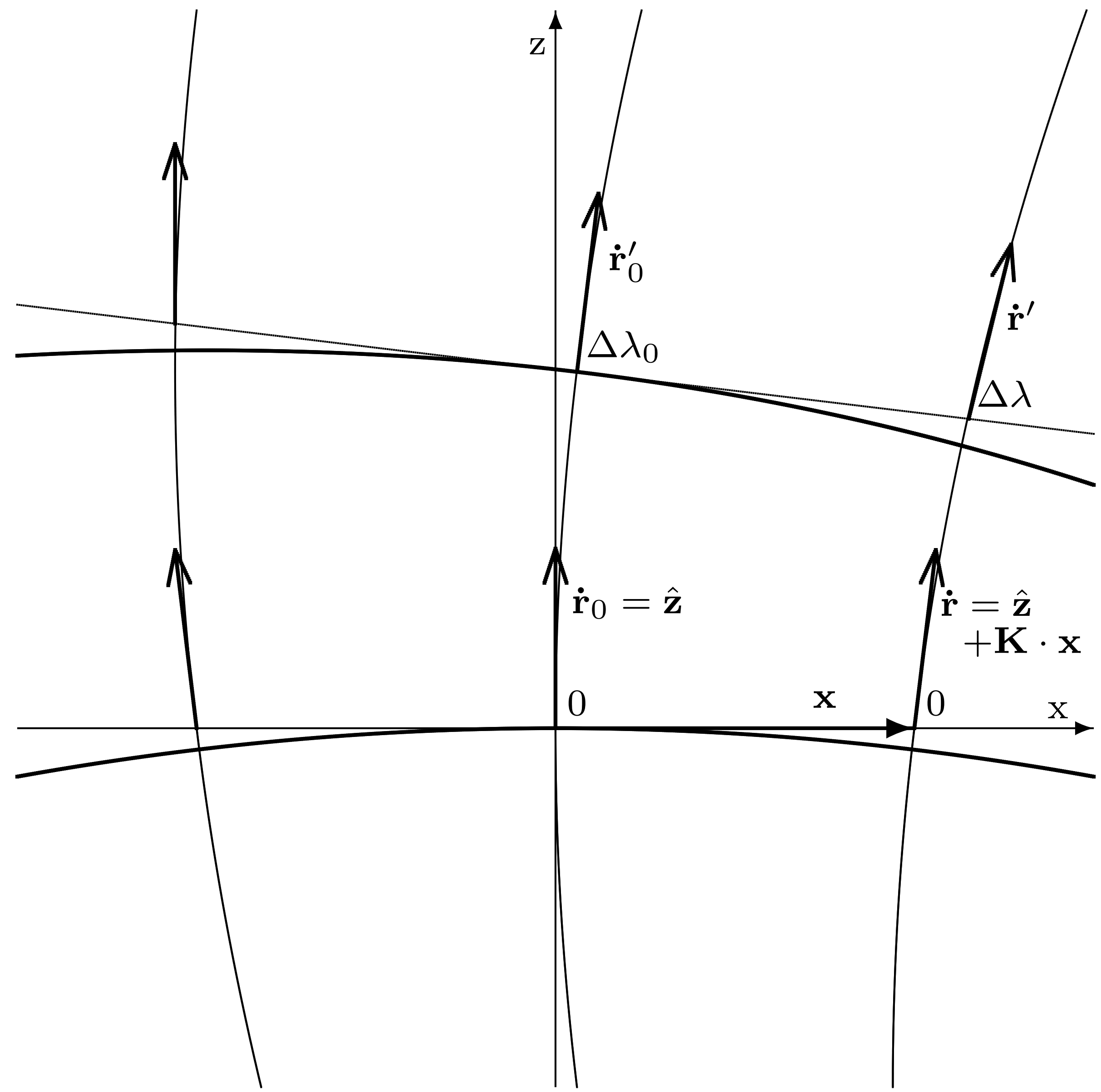


Figure D1. Illustration of a bundle of rays (thin curves) and associated wave-fronts (thick curves) and ray direction vectors $\dot{\mathbf{r}} = d\mathbf{r}/d\lambda$ (arrows). The base of each arrow is labelled by distance (physical for lumpy glass, background conformal for perturbed FRW) along the path. Close to the guiding ray the ray vectors will vary linearly with transverse displacement. The optical tensor $\mathbf{K}$ is the derivative of the ray direction with respect to coordinates $\mathbf{x}$ on the plane that is tangent to the wavefront at the location of the guiding ray. The optical tensor transport equation tells us how $\mathbf{K}$ evolves as the bundle propagates through any metric or refractive index fluctuations. Since rays are perpendicular to the

KP15 on the "*focusing theorem*"? $\langle \ddot{D} \rangle / D_0 = -\langle \Sigma^2 \rangle < 0.$

- We have developed the optical scalar transport equations in a form appropriate when one wishes to specify the metric fluctuations as a stochastic random field (with zero mean for k=0 component)
  - interesting subtlety: one should *not* assume $\langle \delta R \rangle = 0$
  - in inflationary context, small scale space-time curvature fluctuations have to accommodate themselves within the (flat-space) boundary conditions imposed when the larger regions accelerate outside of horizon
- We have solved these to obtain the ensemble average of the perturbation to the *area* of a beam of specified solid angle fired off from the observer and propagating back to the source surface.
- We perform a double expansion, working to second order in δ(metric) and to lowest order in the inverse of "coherence scale"/Hubble scale
- Cancellation: Not just "Born level", but 1st "beyond Born" also
- We were only able to solve for the case where metric fluctuations are non-evolving (like in Einstein - de Sitter) but were able to obtain the "un-focusing theorem": $\langle \Delta A/A \rangle = -2J\lambda/3 + ...$
  - this is consistent with the more general result (variable J) found by more straightforward approach.
- An exactly analogous calculation for $\langle \Delta D/D \rangle$ does not show cancellation and results in much larger ($O(\varkappa^2)$) result. *But just the statistical bias*. QED

Concluding comments....

- The problem of how lensing by cosmic structure affects the mean *distance-redshift relation* (or the mean area of a surface of constant redshift) goes back for at least 50 years
- This is an intellectually compelling problem, and is also potentially important for interpretation of SN1a and CMB observations in the age of "*precision cosmology*"
- A major conflict arose in the '80s between Weinberg's flux conservation argument and the contrary indications from the focussing theorem
- This has remained unresolved and resurfaced recently in results of relativistic 2nd order perturbation theory.

Concluding comments continued...

- NK+Peacock (2015): we have reconciled the conflicts
- We support Weinberg:
  - lensing affects individual source flux densities in a random way
  - but *in the mean* the flux density of sources is *almost exactly unperturbed*
- and Kibble and Lieu
  - who emphasised the distinction between source and direction averaging
- Our main results:
  - Relativistic effects have confused physical effects and statistical biases.
  - there *is* a bias in the area of constant z or photosphere surfaces - but it is very, very small ~ 10^{-6}
  - we have shown that the celebrated "*focusing theorem*", despite its name, does not imply any intrinsic tendency for bundles of rays to be focused as they wend their wiggly way through the lumpy cosmos
- Implication: conventional methods for analysing the CMB & SN1a (mostly) are valid.