Cosmic Acceleration from Modified Gravity:

$f(R)$

A Worked Example

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Why Study $f(R)$?

- Cosmic acceleration, like the cosmological constant, can either be viewed as arising from

  Missing, or dark energy, with $w \equiv \bar{p}/\bar{\rho} < -1/3$

  Modification of gravity on large scales

$$\begin{aligned} G_{\mu\nu} &= 8\pi G \left(T^{\rm M}_{\mu\nu} + T^{\rm DE}_{\mu\nu}\right) \\ F(g_{\mu\nu}) + G_{\mu\nu} &= 8\pi G T^{\rm M}_{\mu\nu} \end{aligned}$$

- Proof of principle models for both exist: quintessence, k-essence; DGP braneworld acceleration, $f(R)$ modified action
- Compelling models for either explanation lacking
- Study models as illustrative toy models whose features can be generalized

Three Regimes

- Three regimes defined by $\gamma = -\Phi/\Psi$ BUT with different dynamics
- Examples $f(R)$ and DGP braneworld acceleration
- Parameterized Post-Friedmann description

- Non-linear regime follows a halo paradigm but a full parameterization still lacking and theoretical, examples few: $f(R)$ now fully worked

General Relativistic Non-Linear Regime	Scalar-Tensor Regime	Conserved-Curvature Regime
	r_*	r_c
halos, galaxy	*large scale structure*	*CMB*

$r \longrightarrow$

Outline

- $f(R)$ Basics and Background
- Linear Theory Predictions
- N-body Simulations and the Chameleon

- Collaborators:
  - Marcos Lima
  - Hiro Oyaizu
  - Hiranya Peiris
  - Iggy Sawicki
  - Fabian Schmidt
  - Yong-Seon Song

$f(R)$ Basics

Cast of $f(R)$ Characters

- R: Ricci scalar or "curvature"
- $f(R)$: modified action (Starobinsky 1980; Carroll et al 2004)

$$S = \int d^4x \sqrt{-g} \left[\frac{R + f(R)}{16\pi G} + \mathcal{L}_{\rm m} \right]$$

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- R: Ricci scalar or "curvature"
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$$S = \int d^4x \sqrt{-g} \left[\frac{R + f(R)}{16\pi G} + \mathcal{L}_{\rm m} \right]$$

- $f_R \equiv df/dR$: additional propagating scalar degree of freedom (metric variation)
- $f_{RR} \equiv d^2f/dR^2$: Compton wavelength of f_R squared, inverse mass squared
- B: Compton wavelength of f_R squared in units of the Hubble length

$$B \equiv \frac{f_{RR}}{1 + f_R} R' \frac{H}{H'}$$

- $' \equiv d/d\ln a$: scale factor as time coordinate

Modified Einstein Equation

- In the Jordan frame, gravity becomes 4th order but matter remains minimally coupled and separately conserved

$$G_{\alpha\beta} + f_R R_{\alpha\beta} - \left(\frac{f}{2} - \Box f_R \right) g_{\alpha\beta} - \nabla_\alpha \nabla_\beta f_R = 8\pi G T_{\alpha\beta}$$

- Trace can be interpreted as a scalar field equation for f_R with a density-dependent effective potential ($p = 0$)

$$3\Box f_R + f_R R - 2f = R - 8\pi G\rho$$

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- For small deviations, $|f_R| \ll 1$ and $|f/R| \ll 1$,

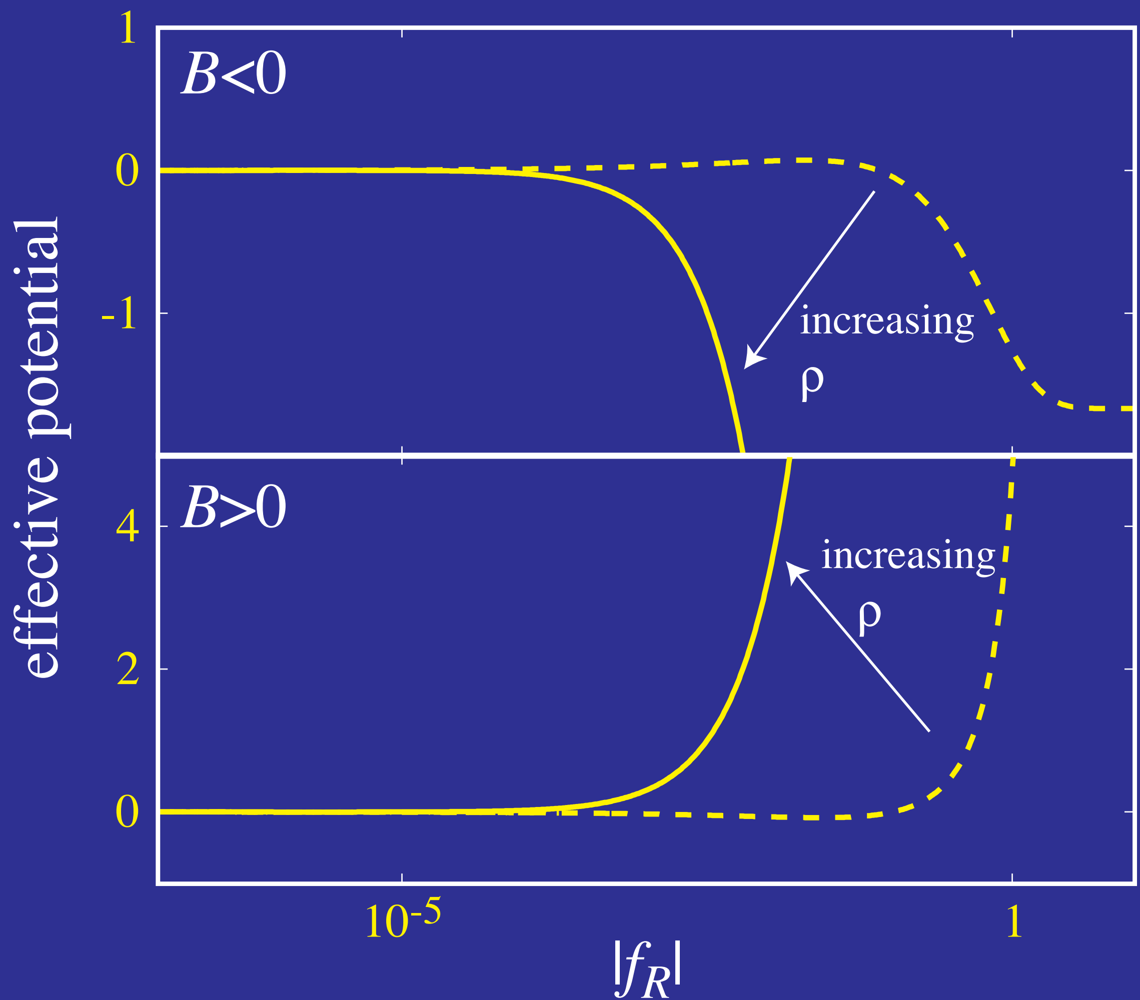
$$\Box f_R \approx \frac{1}{3}(R - 8\pi G\rho)$$

the field is sourced by the deviation from GR relation between curvature and density and has a mass

$$m_{f_R}^2 \approx \frac{1}{3}\frac{\partial R}{\partial f_R} = \frac{1}{3 f_{RR}}$$

Effective Potential

- Scalar f_R rolls in an effective potential that depends on density
- At high density, extrema is at GR $R=8\pi G\rho$
- Minimum for $B>0$, pinning field to $|f_R| \ll 1$, maximum for $B<0$



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$f(R)$ Expansion History

Modified Friedmann Equation

- Expansion history parameterization: Friedmann equation becomes

$$H^2 - f_R(HH' + H^2) + \frac{1}{6}f + H^2 f_{RR} R' = \frac{8\pi G\rho}{3}$$

- Reverse engineering $f(R)$ from the expansion history: for any desired H, solve a 2nd order diffeq to find $f(R)$
- Allows a family of $f(R)$ models, parameterized in terms of the Compton wavelength parameter B

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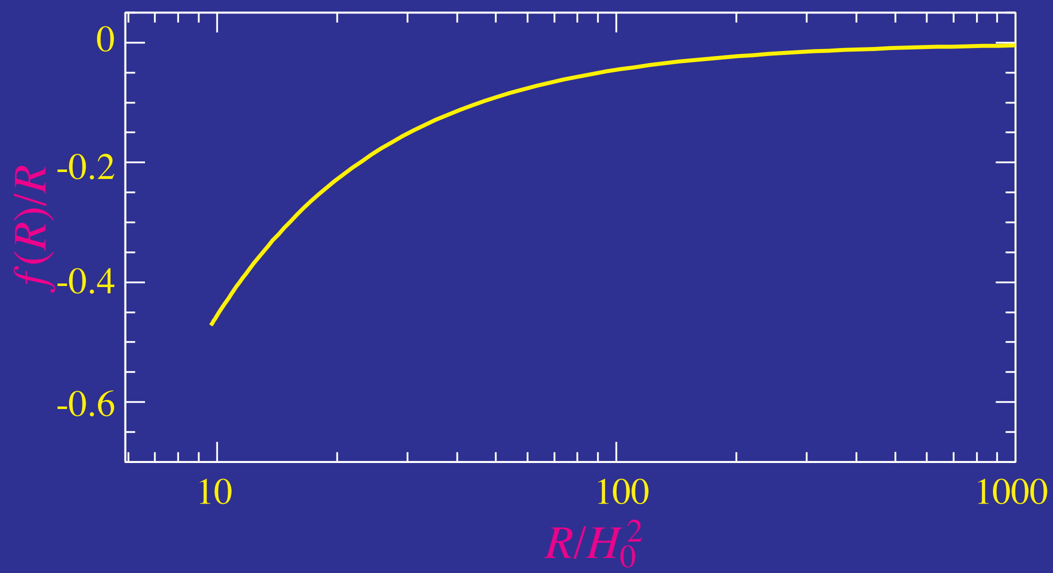
- Reverse engineering $f(R)$ from the expansion history: for any desired H, solve a 2nd order diffeq to find $f(R)$
- Allows a family of $f(R)$ models, parameterized in terms of the Compton wavelength parameter B
- Formally includes models where $B < 0$, such as $f(R) = -\mu^4/R$, leading to confusion as to whether such models provide viable expansion histories
- Answer: no these have short-time scale tachyonic instabilities at high curvature and limit as $B \to 0$ from below is not GR
- $B > 0$ family has very different implications for structure formation but with identical distance-redshift relations

Expansion History Family of $f(R)$

- Each expansion history, matched by dark energy model $[w(z),\Omega_{DE},H_0]$ corresponds to a family of $f(R)$ models due to its 4th order nature
- Parameterized by $B \propto f_{RR} = d^2f/dR^2$ evaluated at z=0



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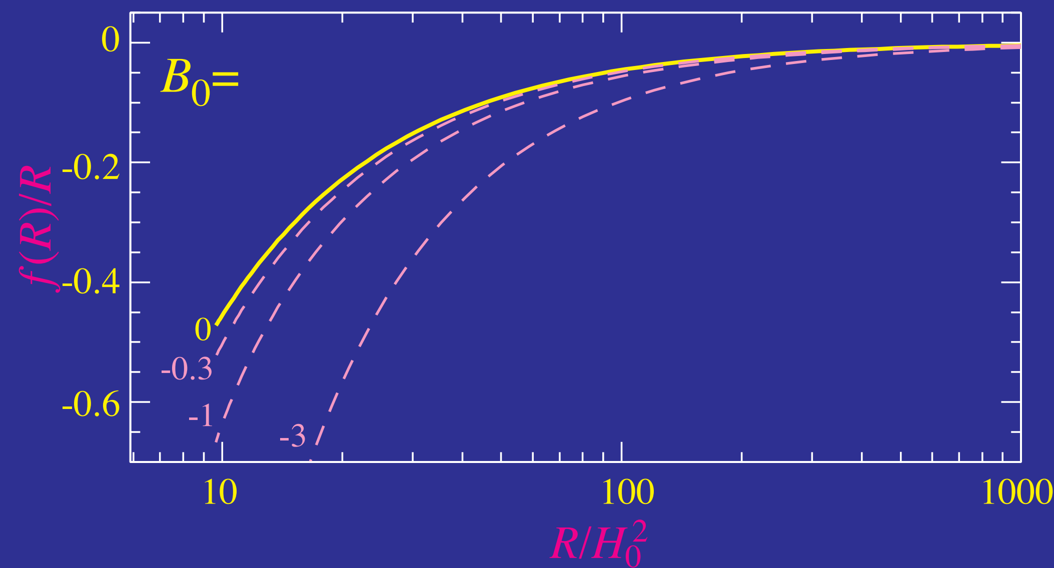

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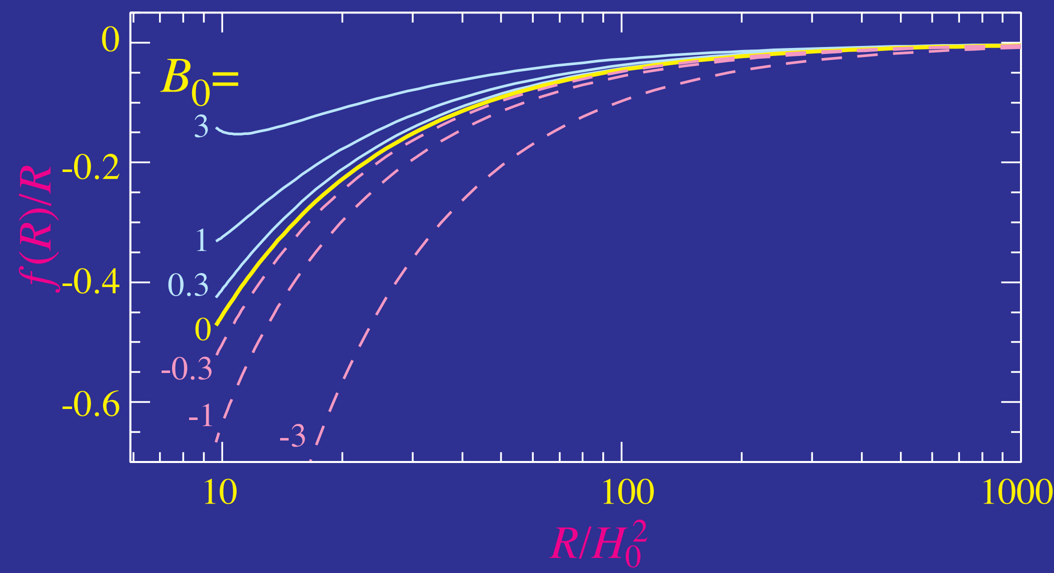

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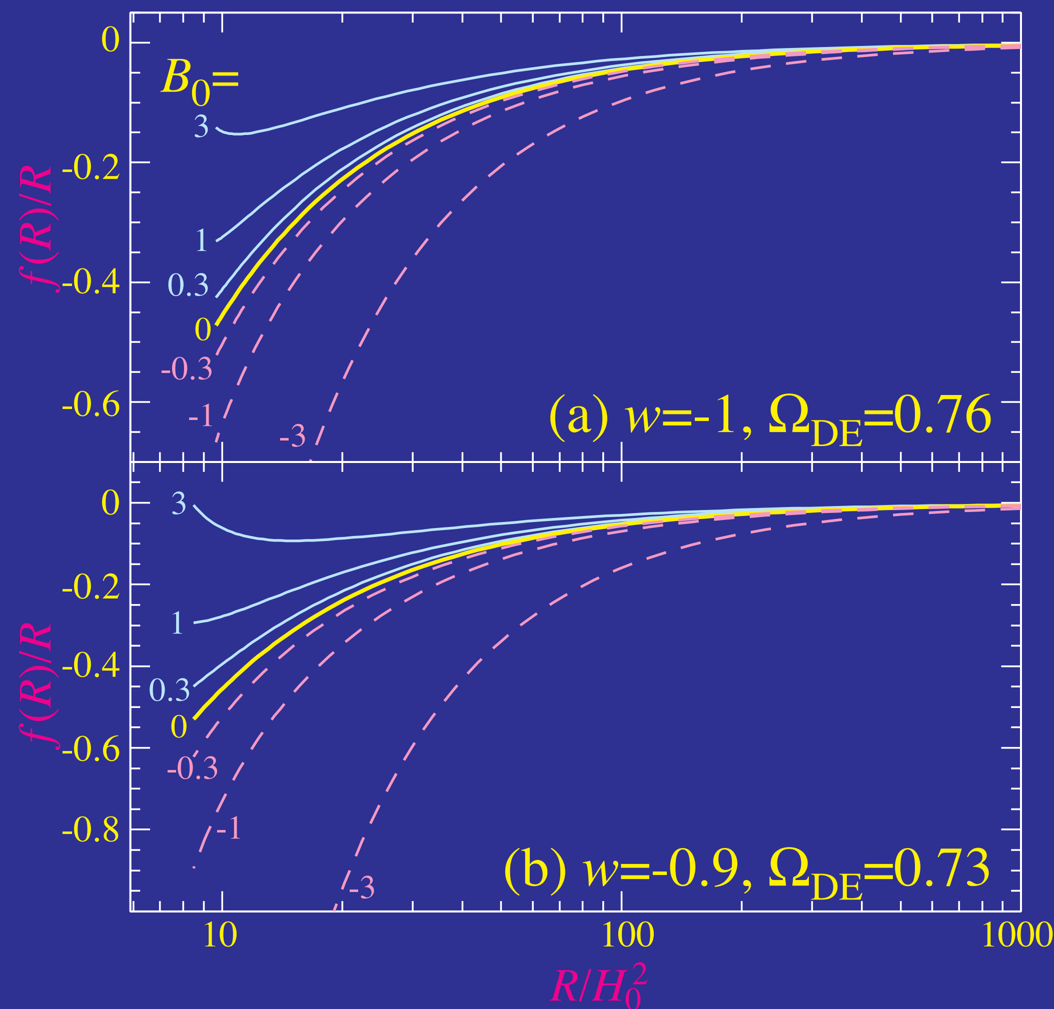

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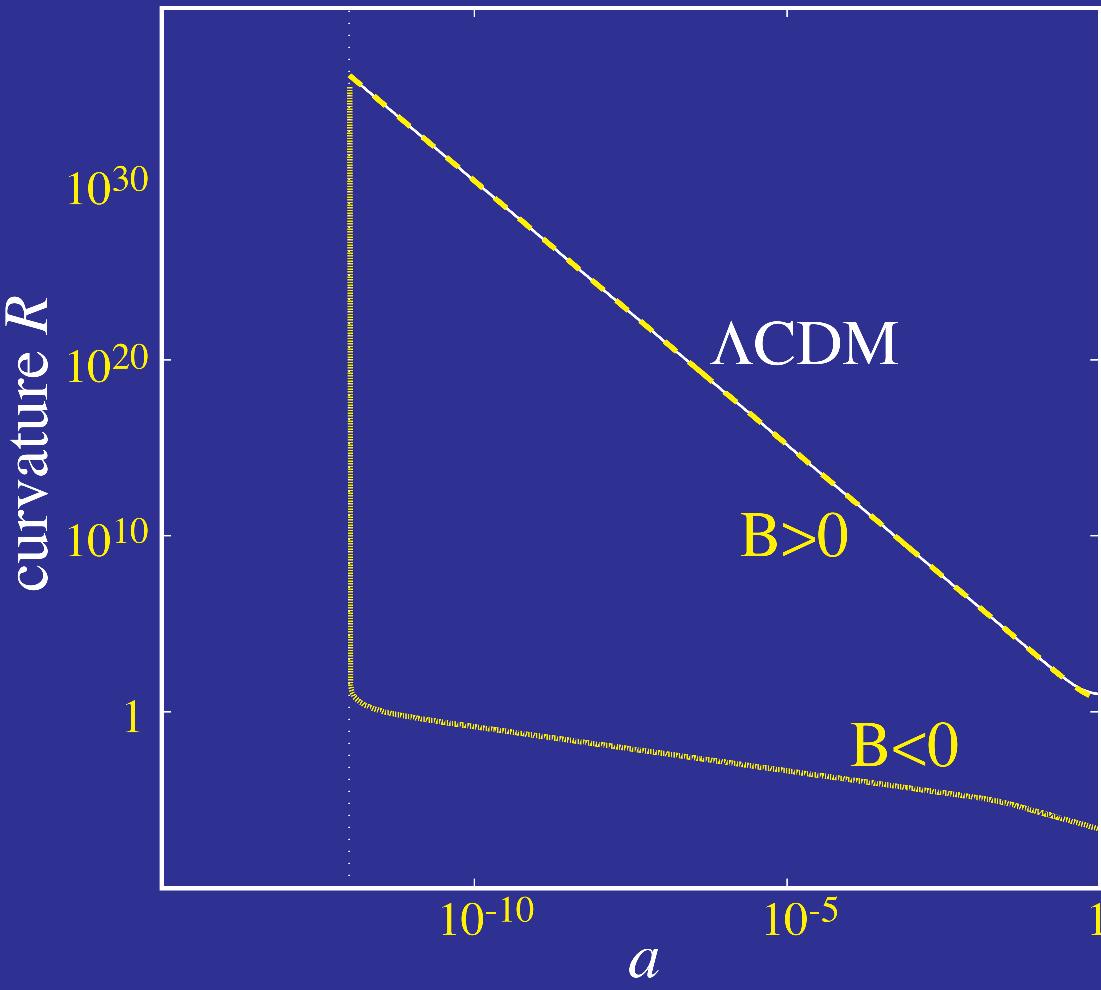
Instability at High Curvature

- Tachyonic instability for negative mass squared $B<0$ makes high curvature regime increasingly unstable: high density ≠ high curvature
- Linear metric perturbations immediately drop the expansion history to low curvature solution



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$f(R)$ Linear Theory

Three Regimes

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r_* $\quad$ r_c

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$r \longrightarrow$

Curvature Conservation

- On superhorizon scales, energy momentum conservation and expansion history constrain the evolution of metric fluctuations (Bertschinger 2006)
- For adiabatic perturbations in a flat universe, conservation of comoving curvature applies $\zeta' = 0$ where $' \equiv d/d\ln a$ (Bardeen 1980)
- Gauge transformation to Newtonian gauge

$$ds^2 = -(1+2\Psi)dt^2 + a^2(1+2\Phi)dx^2$$

yields (Hu & Eisenstein 1999)

$$\Phi'' - \Psi' - \frac{H''}{H'}\Phi' - \left(\frac{H'}{H} - \frac{H''}{H'}\right)\Psi = 0$$

- Modified gravity theory supplies the closure relationship $\Phi = -\gamma(\ln a)\Psi$ between and expansion history $H = \dot{a}/a$ supplies rest.

Linear Theory for $f(R)$

- In $f(R)$ model, "superhorizon" behavior persists until Compton wavelength smaller than fluctuation wavelength $B^{1/2}(k/aH) < 1$
- Once Compton wavelength becomes larger than fluctuation

$$B^{1/2}(k/aH) > 1$$

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- Small scale density growth enhanced and

$$8\pi G\rho > R$$

low curvature regime with order unity deviations from GR
- Transitions in the non-linear regime where the Compton wavelength can shrink via chameleon mechanism
- Given $k_{\rm NL}/aH \gg 1$, even very small f_R have scalar-tensor regime

Deviation Parameter

- Express the 4th order nature of equations as a deviation parameter

$$\Phi'' - \Psi' - \frac{H''}{H'}\Phi' - \left(\frac{H'}{H} - \frac{H''}{H'}\right)\Psi = \left(\frac{k}{aH}\right)^2 B\epsilon$$

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- In high redshift, high curvature R limit this is

$$\epsilon'' + \left(\frac{7}{2} + 4\frac{B'}{B}\right)\epsilon' + \frac{2}{B}\epsilon = \frac{1}{B} \times \text{ metric sources}$$

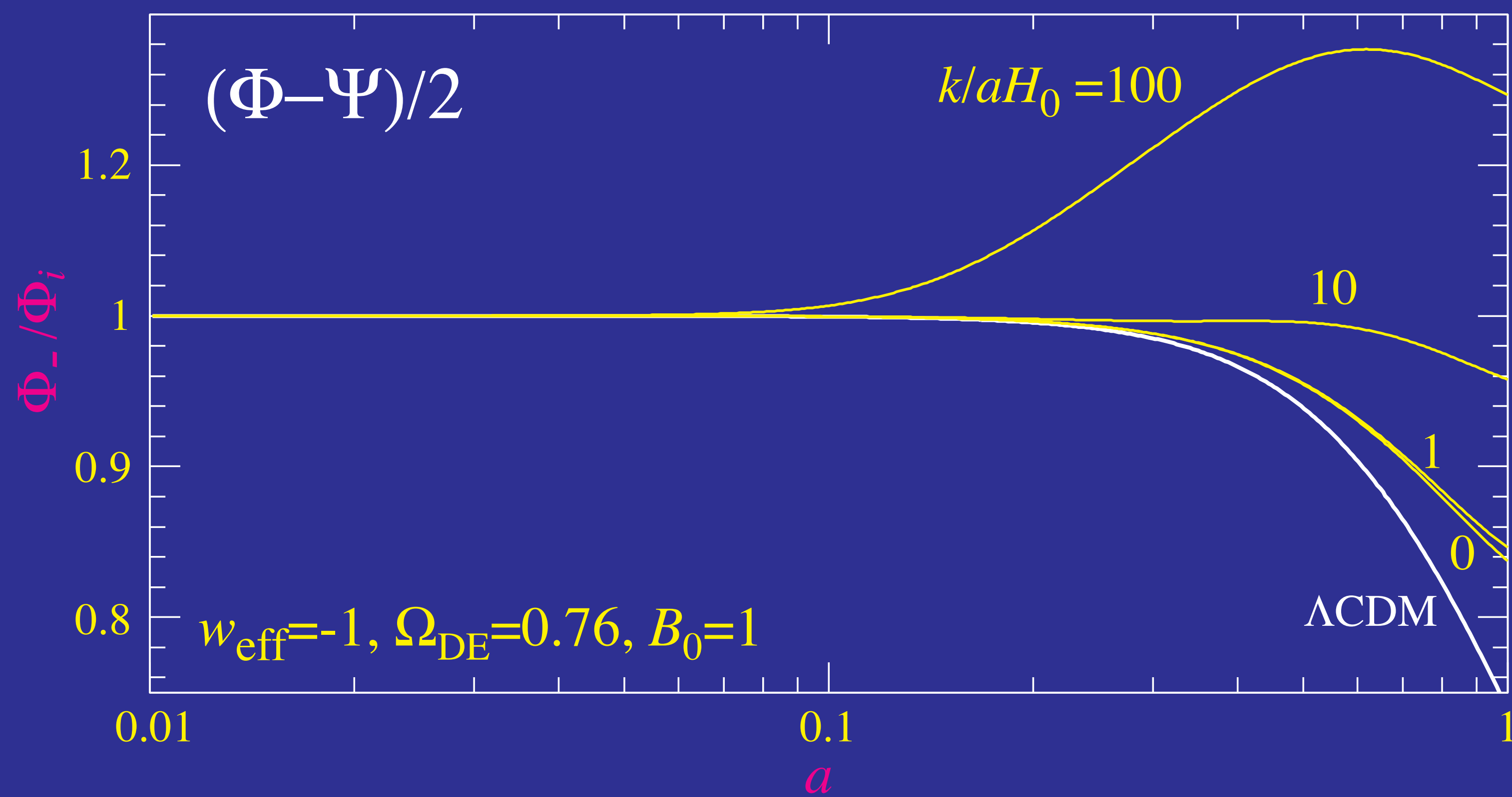
$$B = \frac{f_{RR}}{1+f_R} R' \frac{H}{H'}$$

- $R \to \infty$, $B \to 0$ and for $B < 0$ short time-scale tachyonic instability appears making previous models not cosmologically viable

$$f(R) = -M^{2+2n}/R^n$$

Potential Growth

- On the stable $B>0$ branch, potential evolution reverses from decay to growth as wavelength becomes smaller than Compton scale
- Quasistatic equilibrium reached in linear theory with $\gamma=-\Phi/\Psi=1/2$ until non-linear effects restore $\gamma=1$



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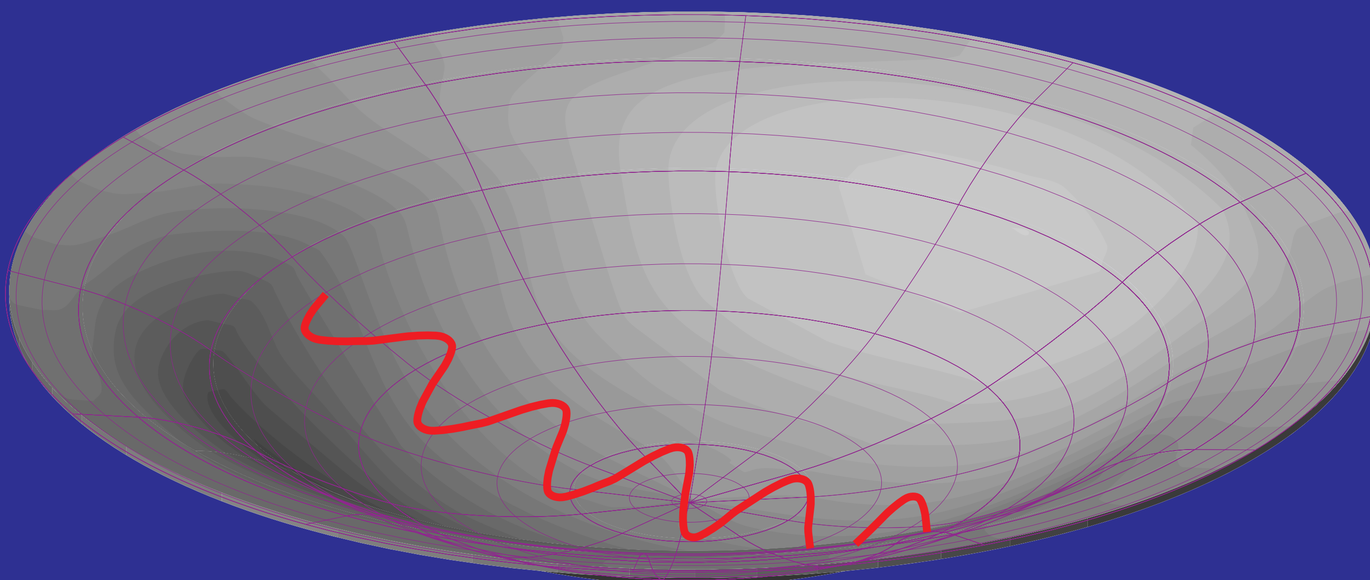
Integrated Sachs-Wolfe Effect

- CMB photons transit gravitational potentials of large-scale structure
- If potential decays during transit, gravitational blueshift of infall not cancelled by gravitational redshift of exit
- Spatial curvature of gravitational potential leads to additional effect $\Delta T/T = -\Delta(\Phi - \Psi)$



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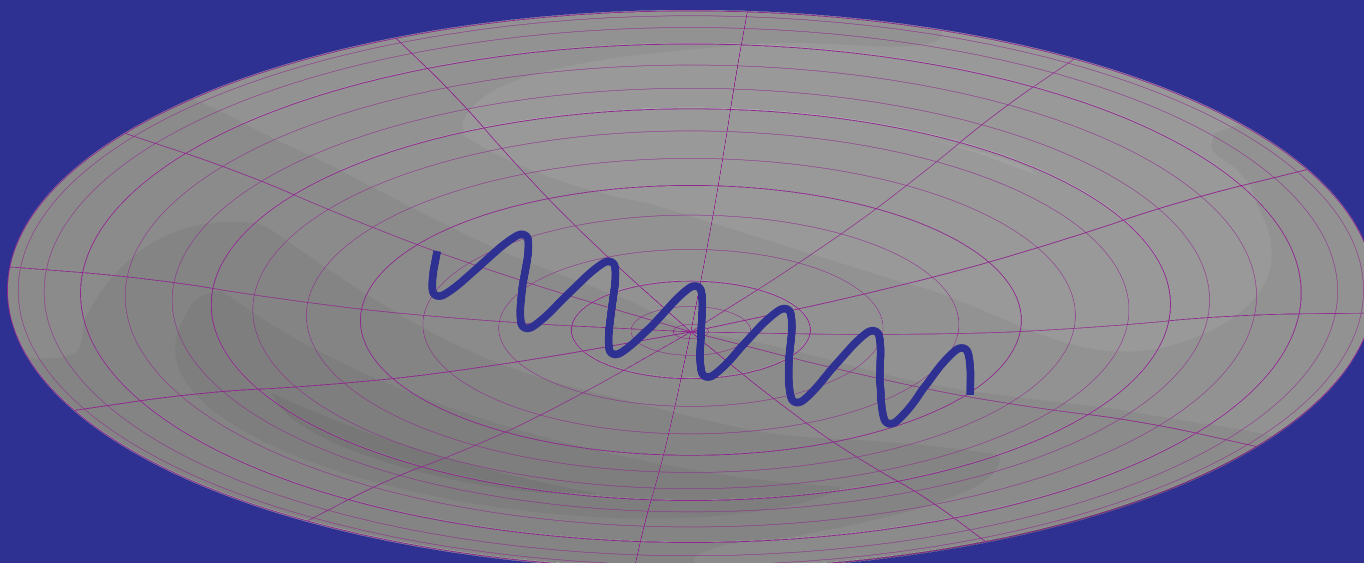
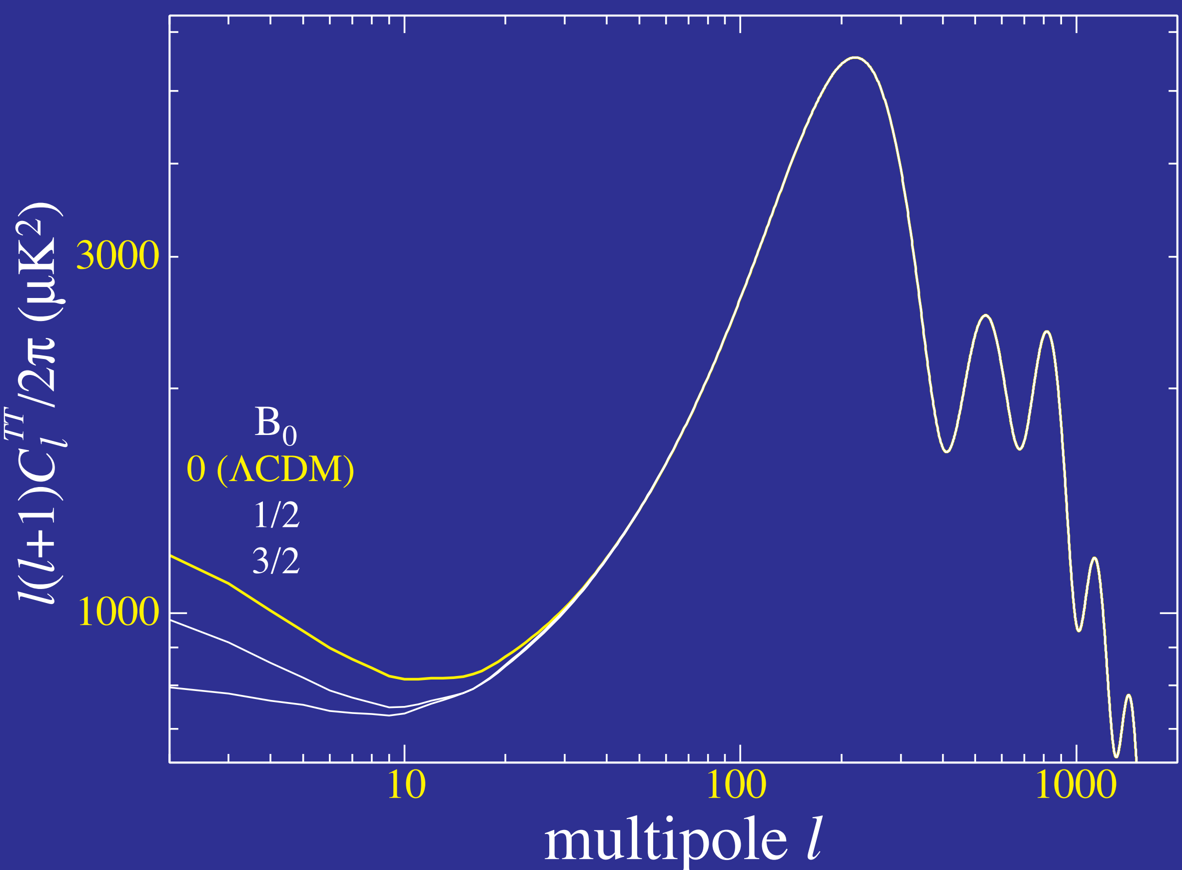


ISW Quadrupole

- Reduction of large angle anisotropy for B_0~1 for same expansion history and distances as ΛCDM
- Well-tested small scale anisotropy unchanged



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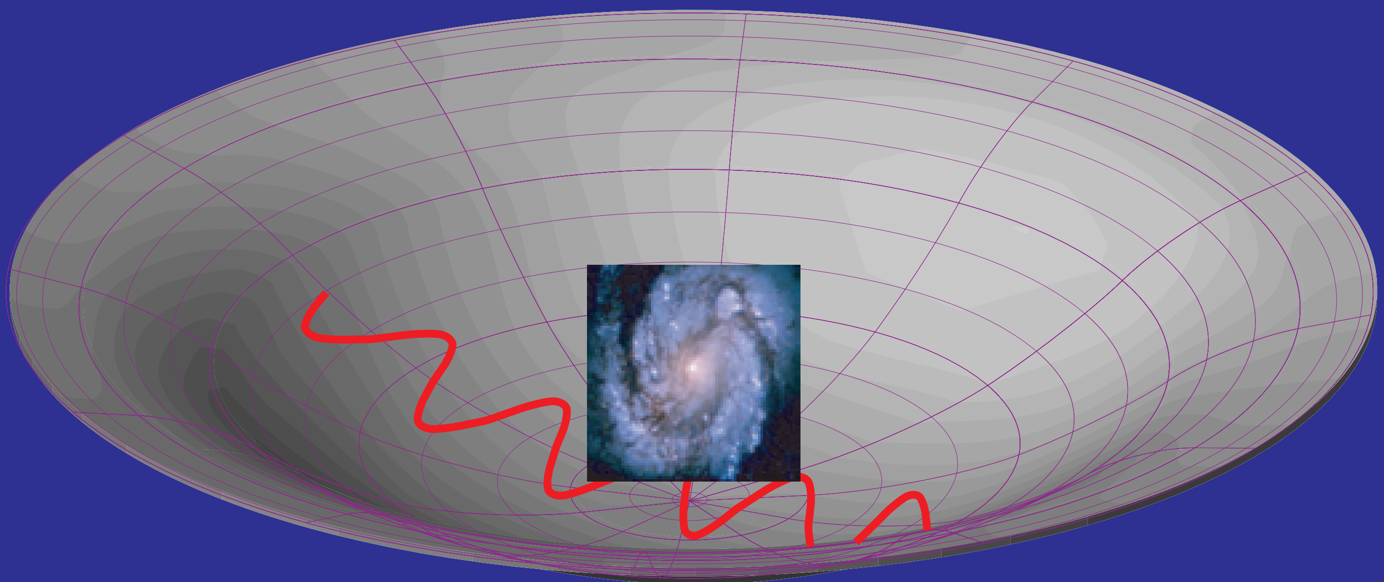
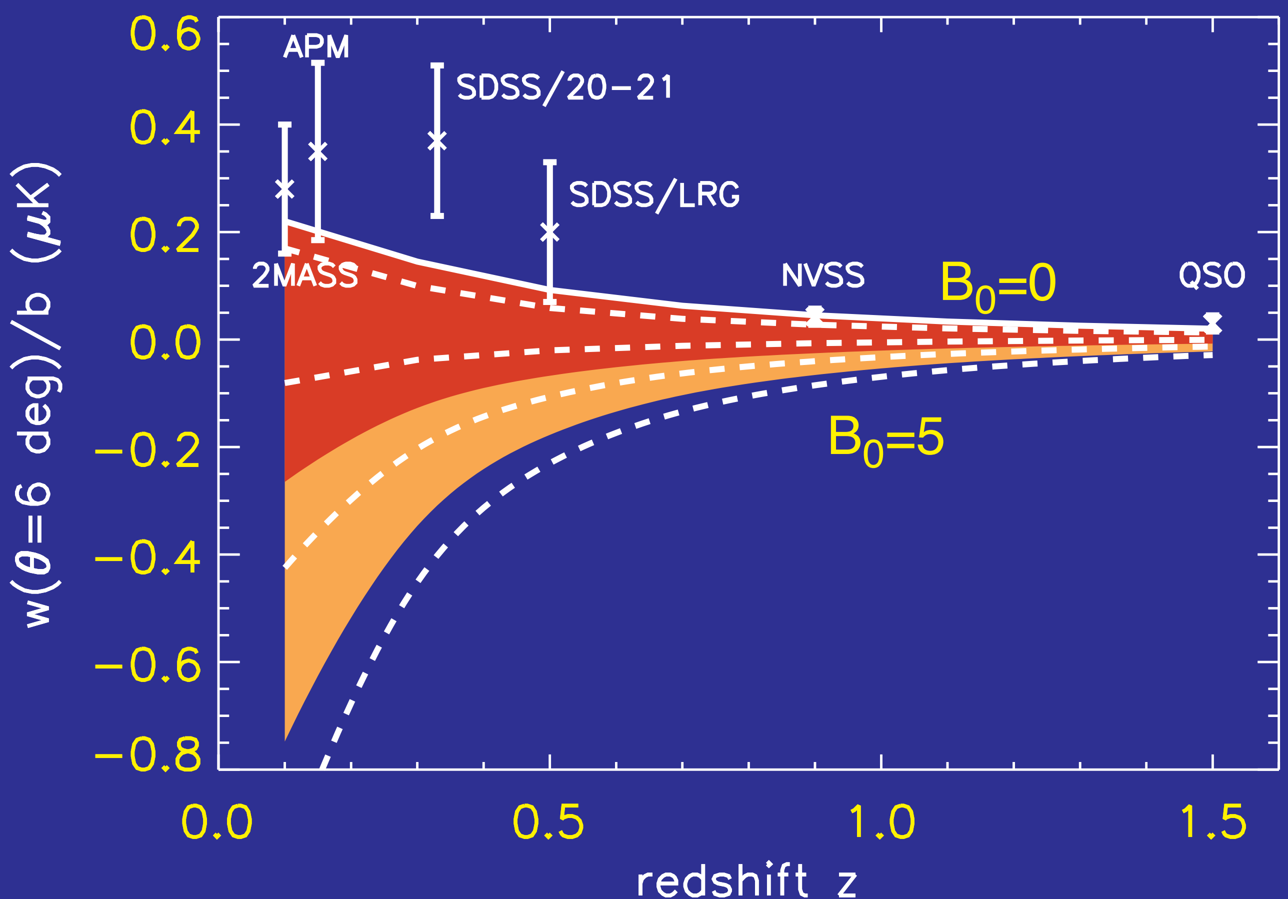
ISW-Galaxy Correlation

- Decaying potential: galaxy positions correlated with CMB
- Growing potential: galaxy positions anticorrelated with CMB
- Observations indicate correlation



Galaxy-ISW Anti-Correlation

- Large Compton wavelength $B^{1/2}$ creates potential growth which can anti-correlate galaxies and the CMB
- In tension with detections of positive correlations across a range of redshifts



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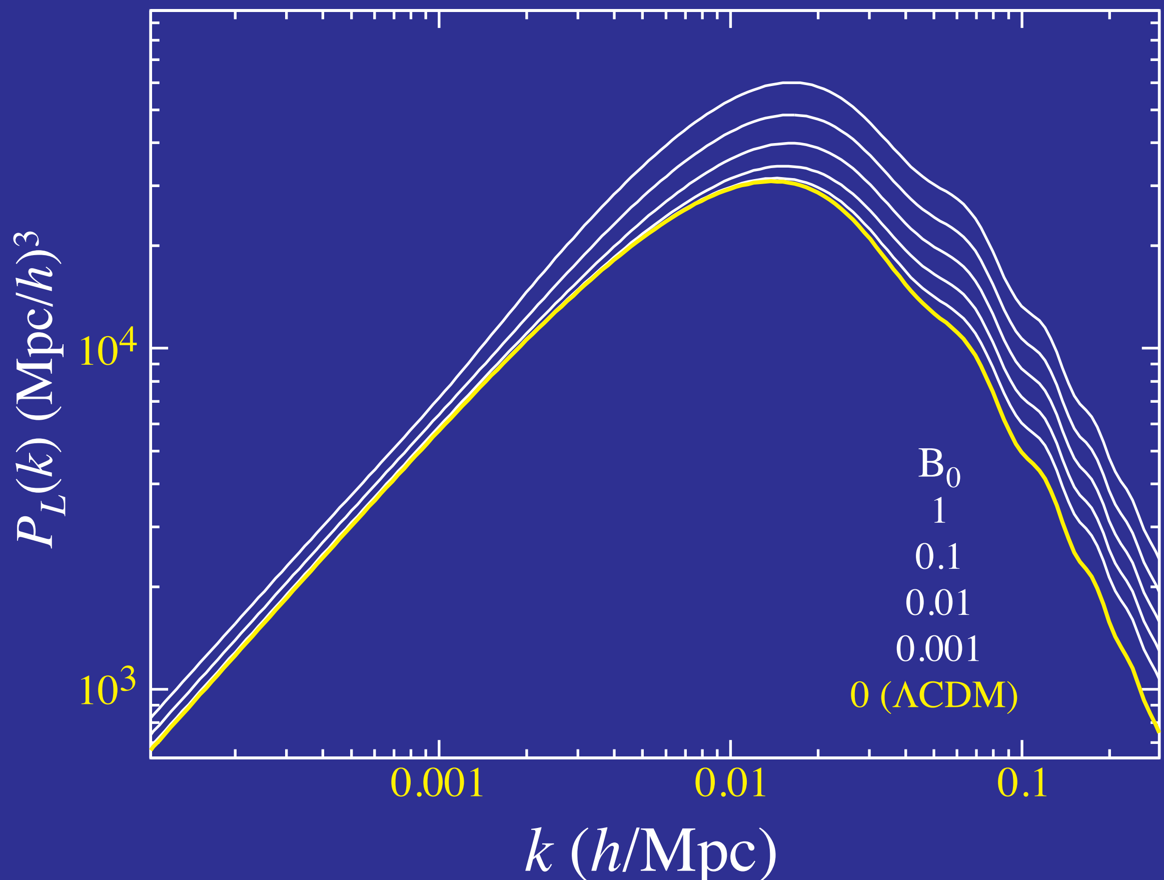
Linear Power Spectrum

- Linear real space power spectrum enhanced on scales below Compton scale in the background
- Scale-dependent growth rate and potentially large deviations on small scales



$f(R)$ Non-Linear Evolution

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r_*

Scalar-Tensor Regime

r_c

Conserved-Curvature Regime

halos, galaxy

large scale structure

CMB

$r \longrightarrow$

Non-Linear Chameleon

- For $f(R)$ the field equation
$$\nabla^2 f_R \approx \frac{1}{3}(\delta R(f_R) - 8\pi G \delta\rho)$$
is the non-linear equation that returns general relativity
- High curvature implies short Compton wavelength and suppressed deviations but requires a change in the field from the background value $\delta R(f_R)$
- Change in field is generated by density perturbations just like gravitational potential so that the chameleon appears only if
$$\Delta f_R \le \frac{2}{3}\Phi\,,$$
else required field gradients too large despite $\delta R = 8\pi G \delta\rho$ being the local minimum of effective potential

Non-Linear Dynamics

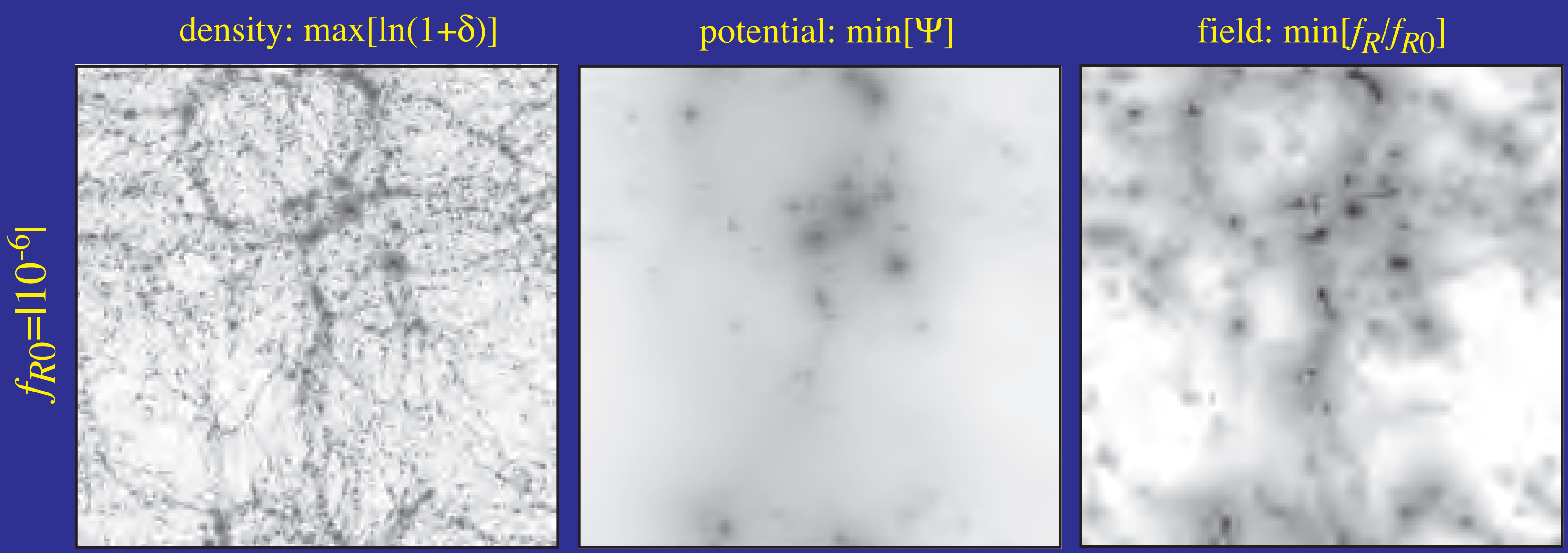
- Supplement that with the modified Poisson equation

$$\nabla^2\Psi = \frac{16\pi G}{3}\delta\rho - \frac{1}{6}\delta R(f_R)$$

- Matter evolution given metric unchanged: usual motion of matter in a gravitational potential Ψ
- Prescription for N-body code
- Particle Mesh (PM) for the Poisson equation
- Field equation is a non-linear Poisson equation: relaxation method for f_R
- Initial conditions set to GR at high redshift

Environment Dependent Force

- Chameleon suppresses extra force (scalar field) in high density, deep potential regions



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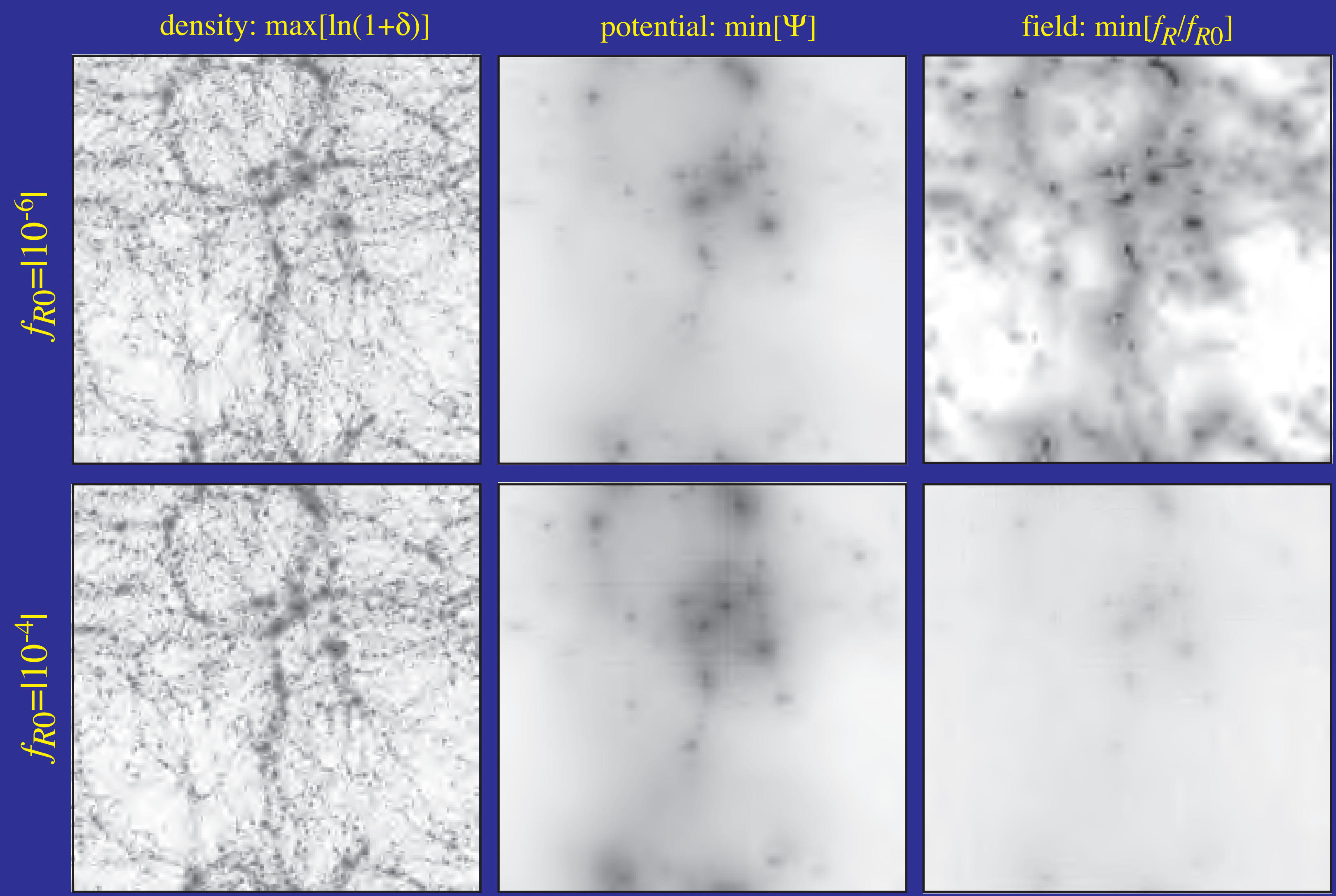
Environment Dependent Force

- For large background field, gradients in the scalar prevent the chameleon from appearing



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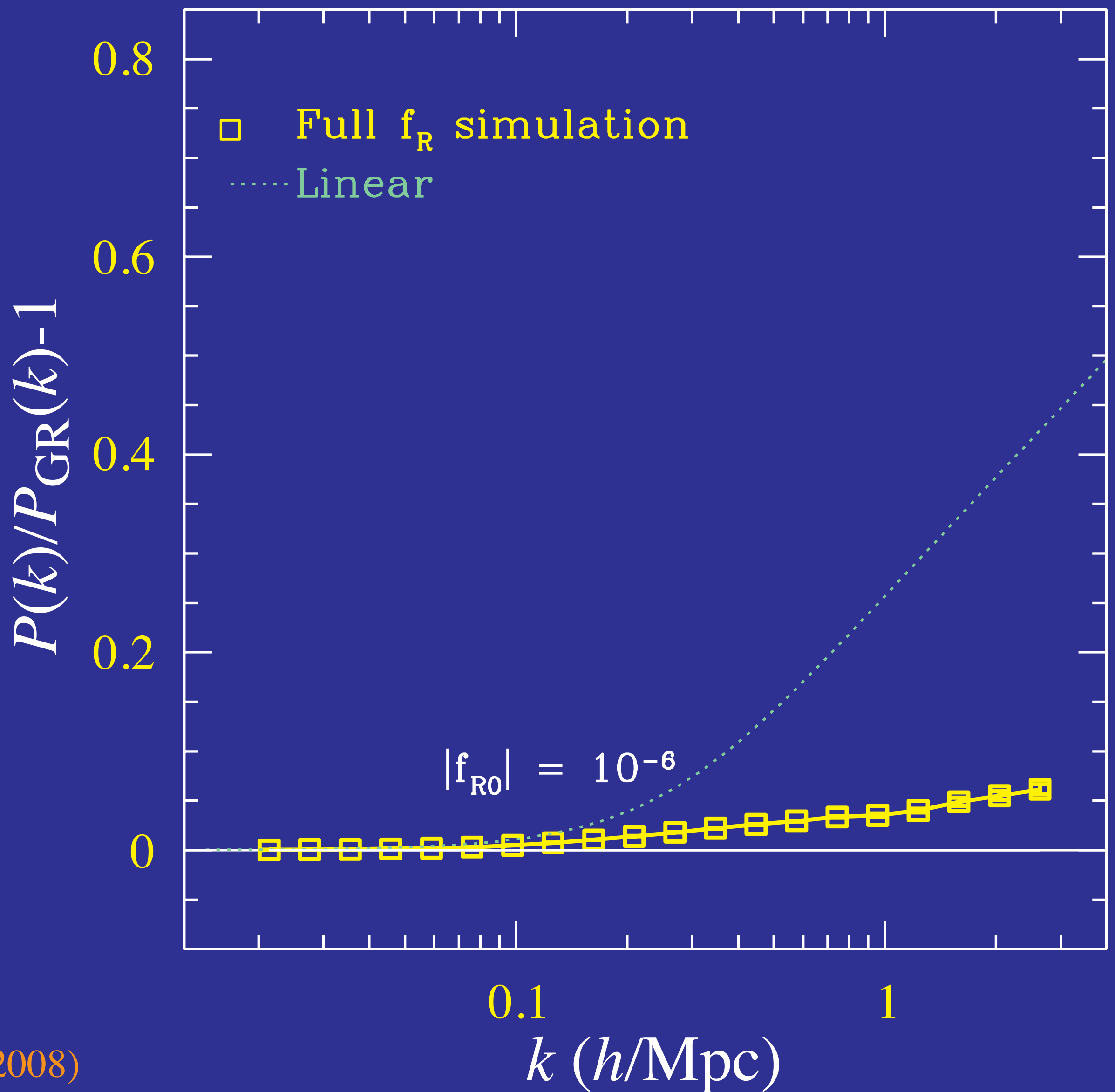
N-body Power Spectrum

- 512^3 PM-relaxation code resolves the chameleon transition to GR: greatly reduced non-linear effect



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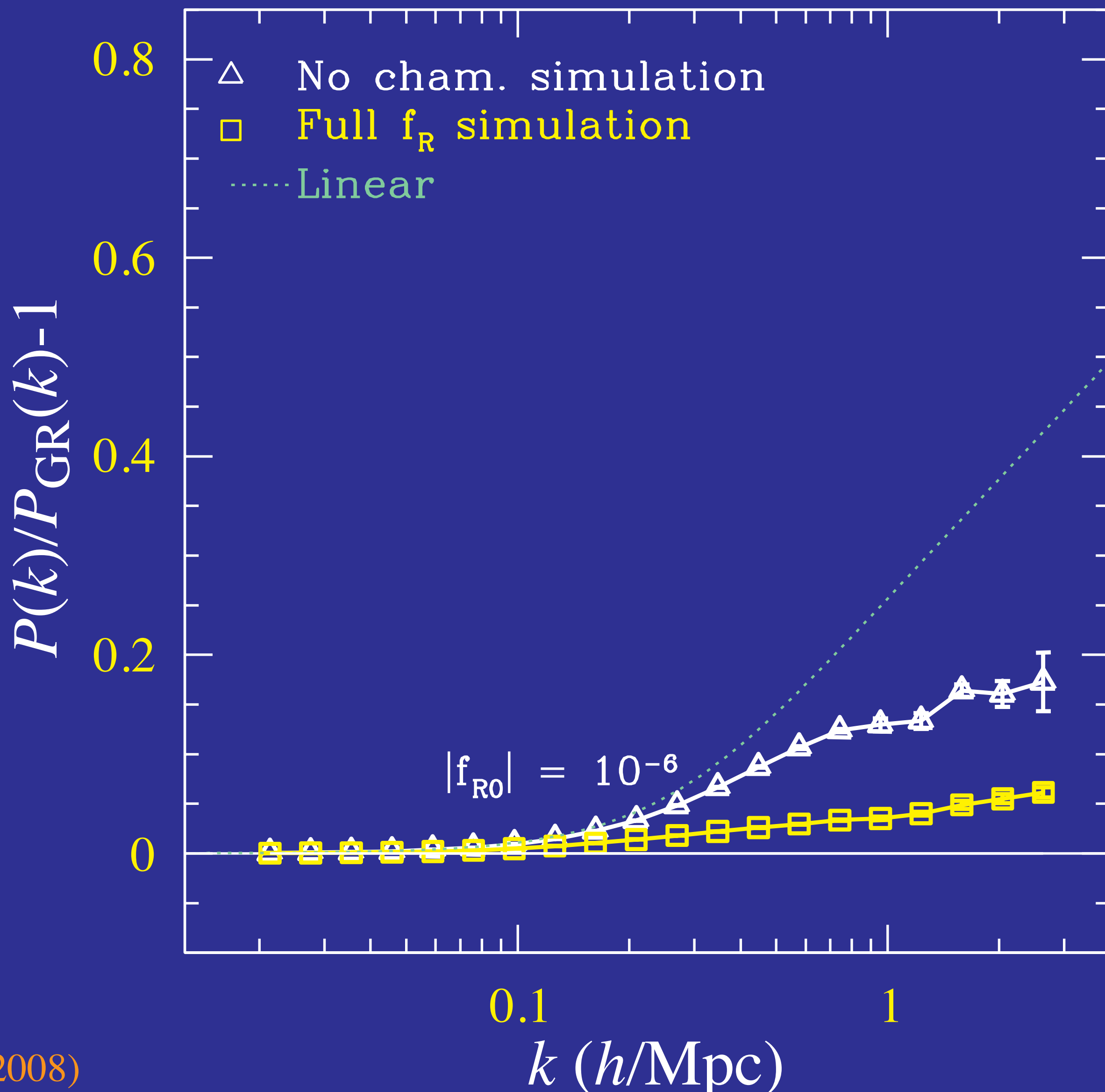
N-body Power Spectrum

- Artificially turning off the chameleon mechanism restores much of enhancement



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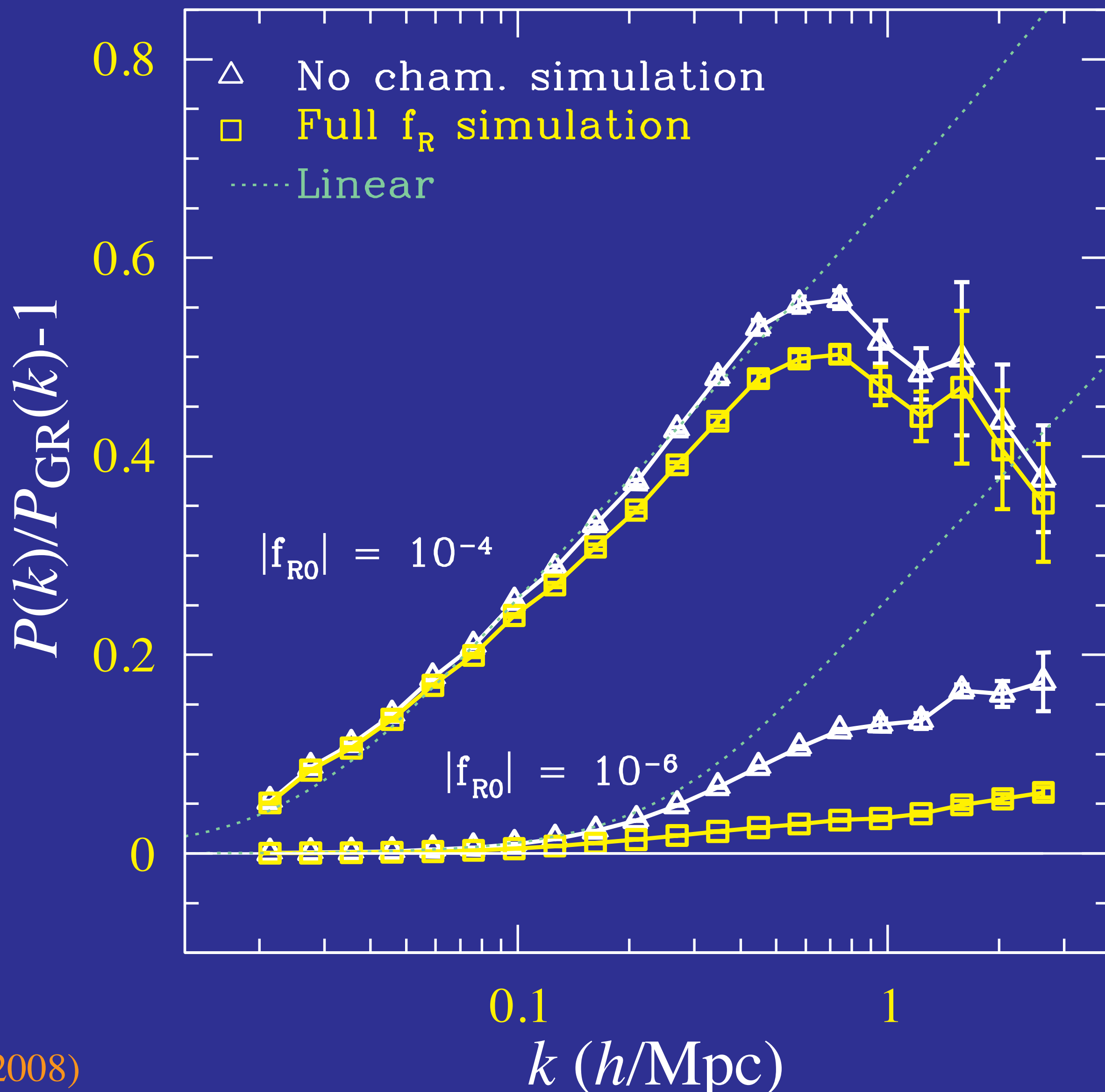
N-body Power Spectrum

- Models where the chameleon absent today (large field models) show residual effects from a high redshift chameleon



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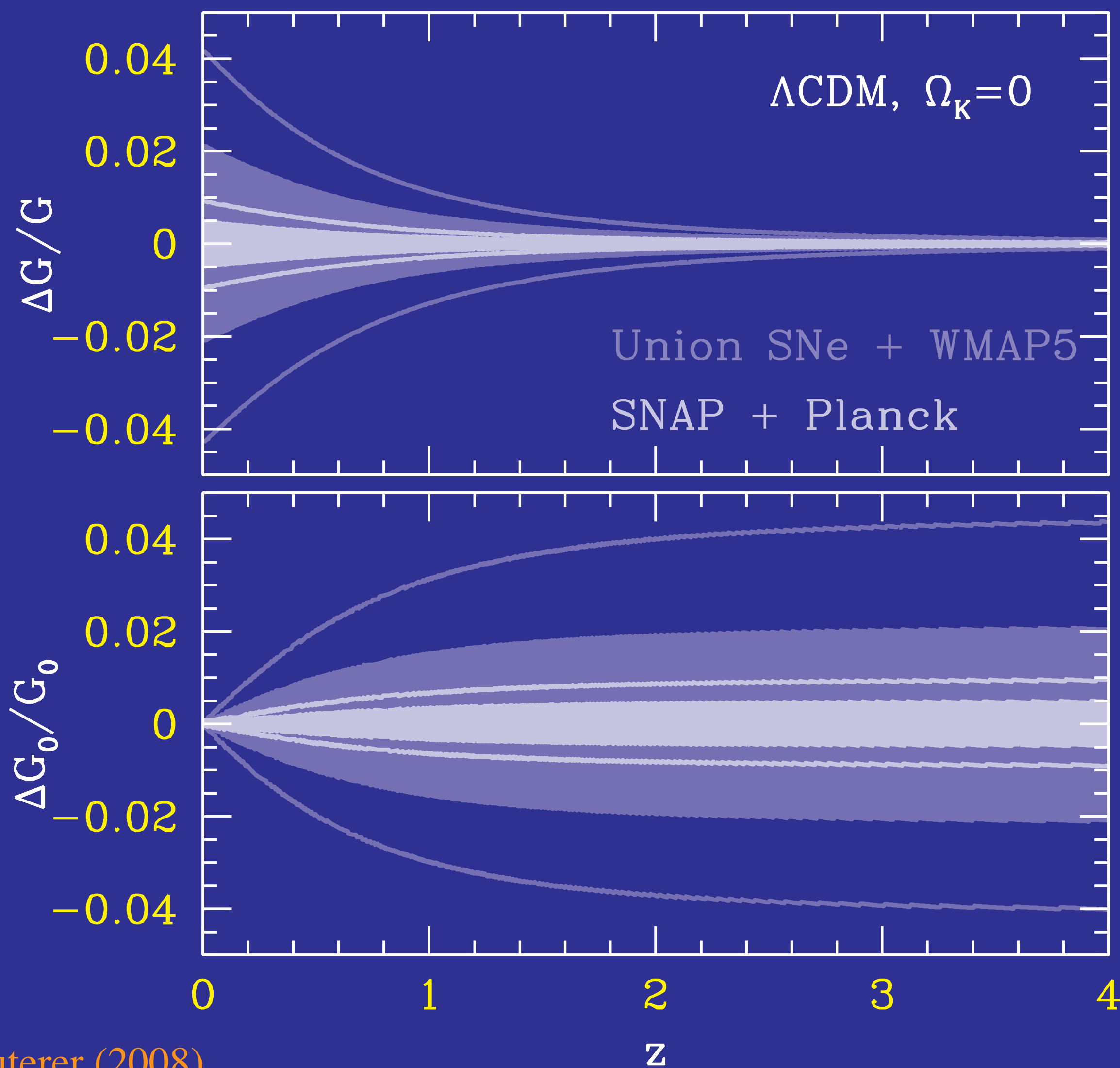
Distance Predicts Growth

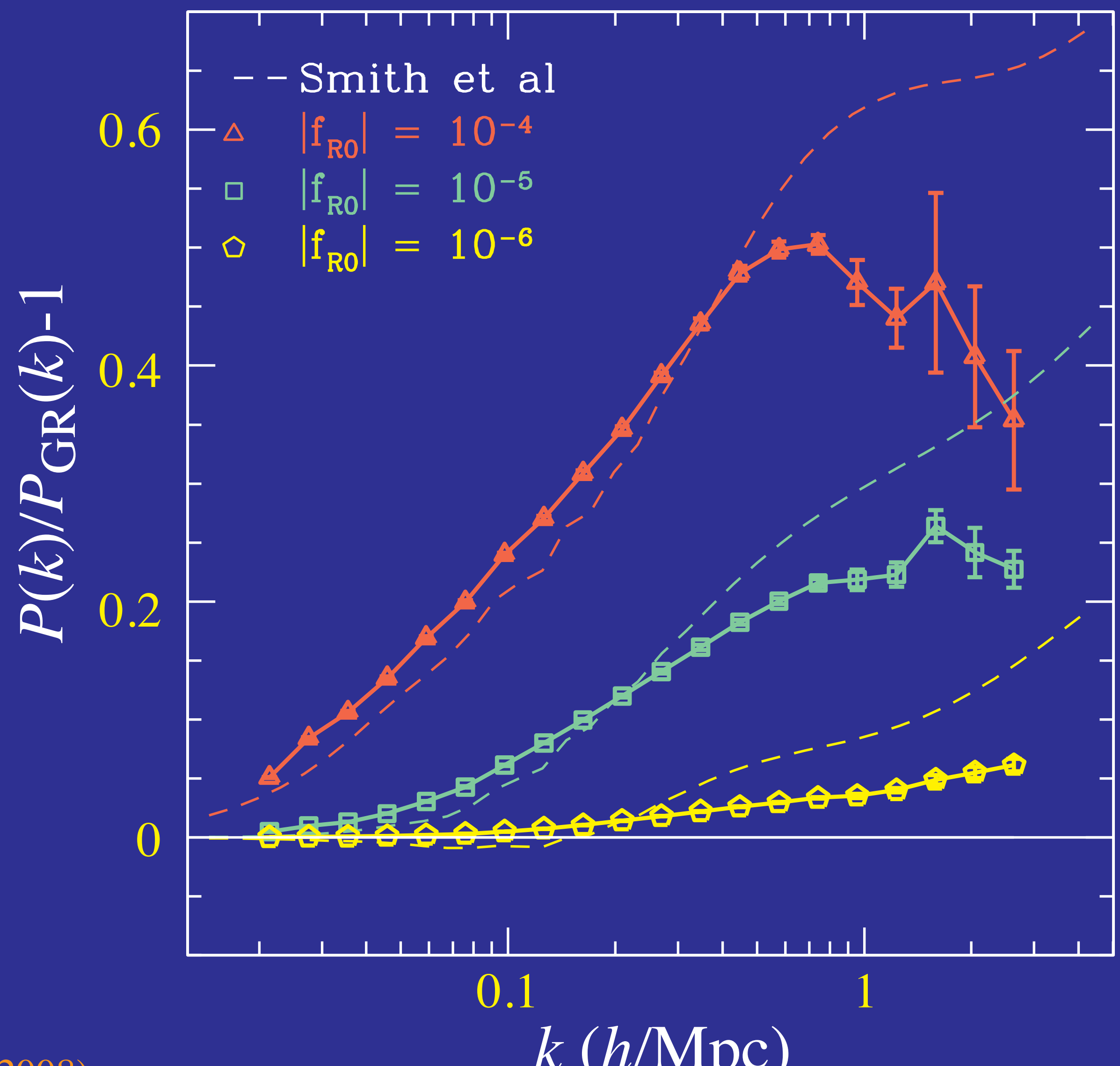
- With smooth dark energy, distance predicts scale-invariant growth to a few percent - a falsifiable prediction



Mortonson, Hu, Huterer (2008)

Scaling Relations

- Fitting functions based on normal gravity fail to capture chameleon and effect of extra forces on dark matter halos



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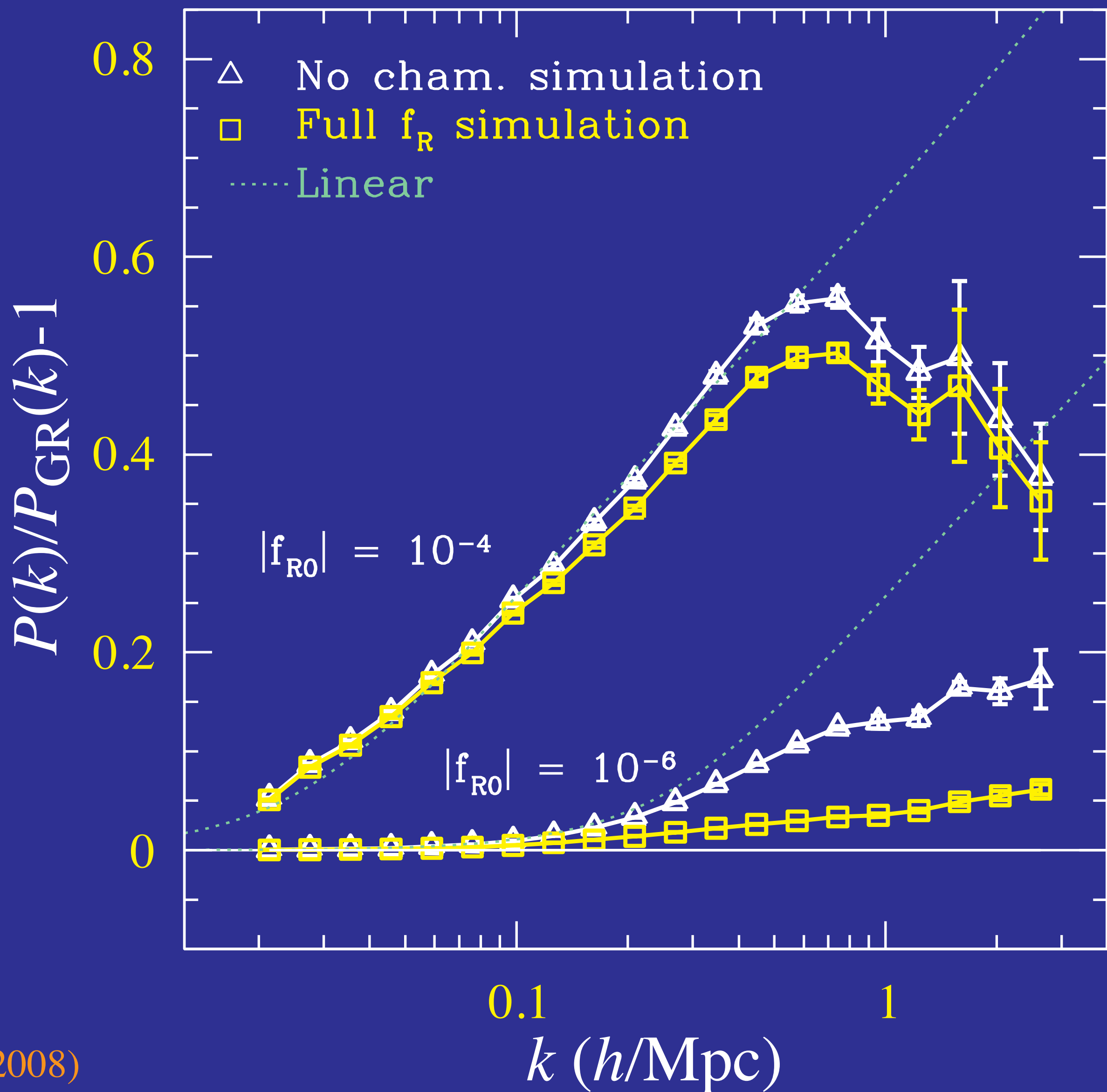

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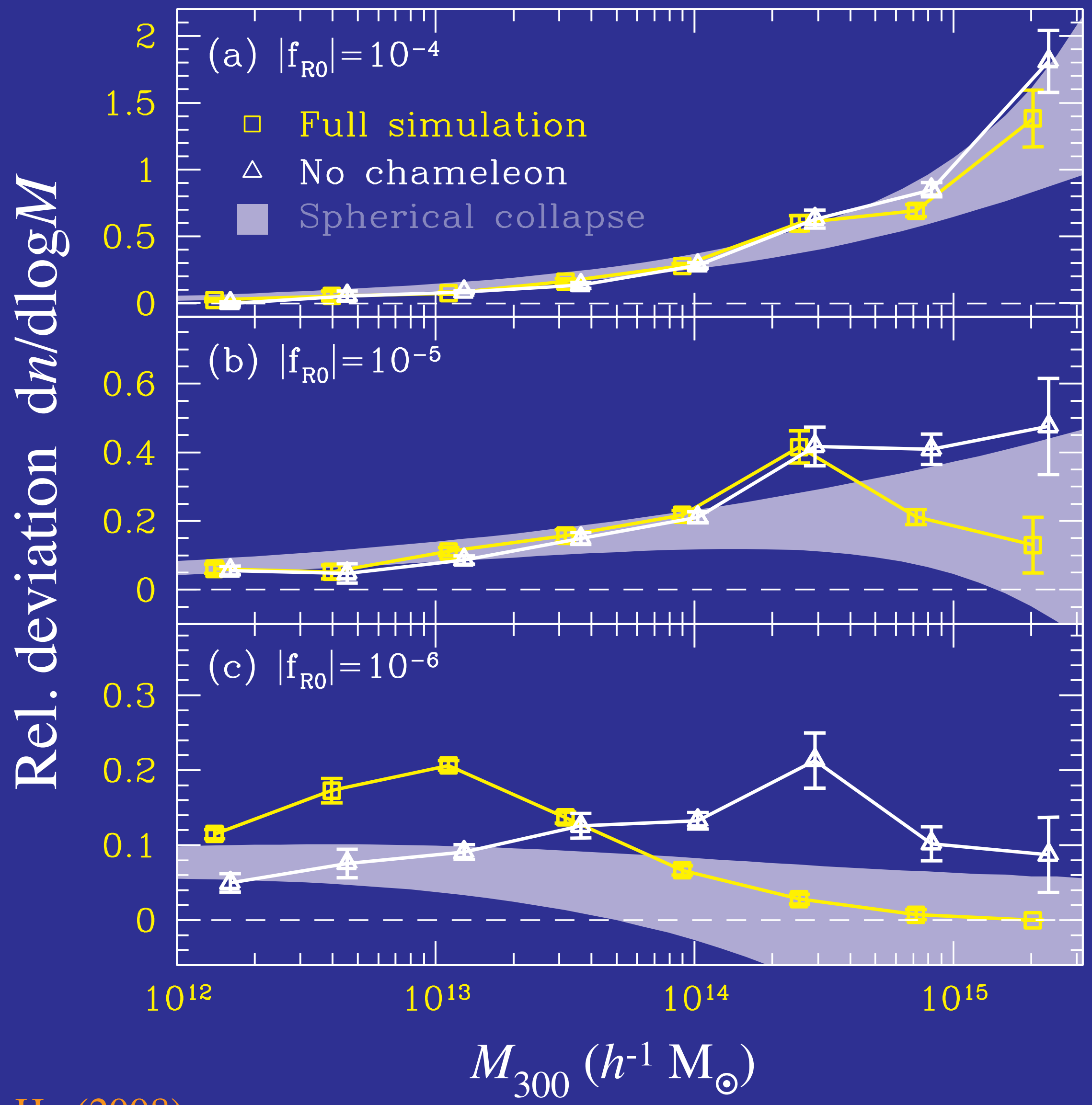
Mass Function

- Enhanced abundance of rare dark matter halos (clusters) with extra force



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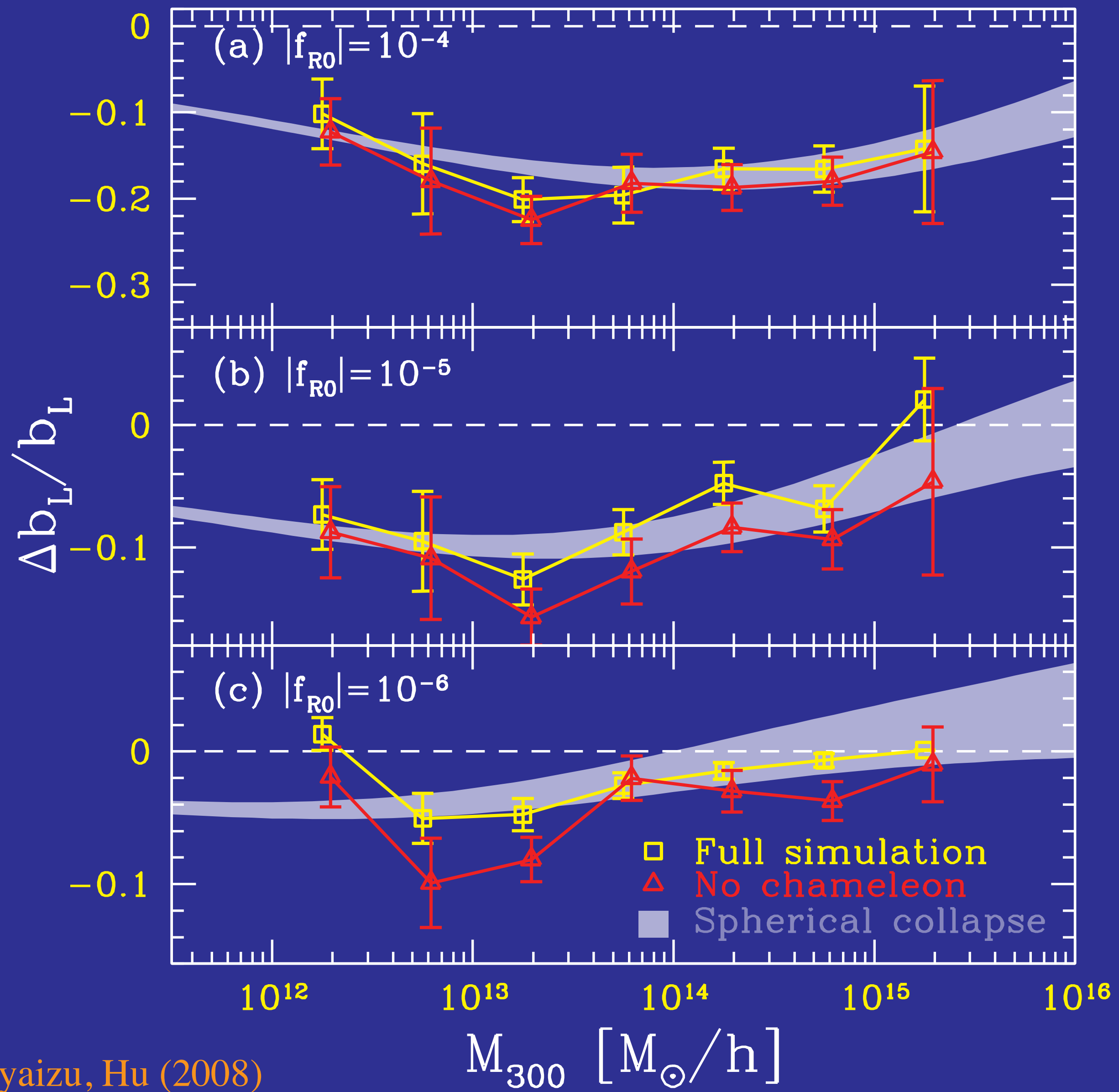
Halo Bias

- Halos at a fixed mass less rare and less highly biased



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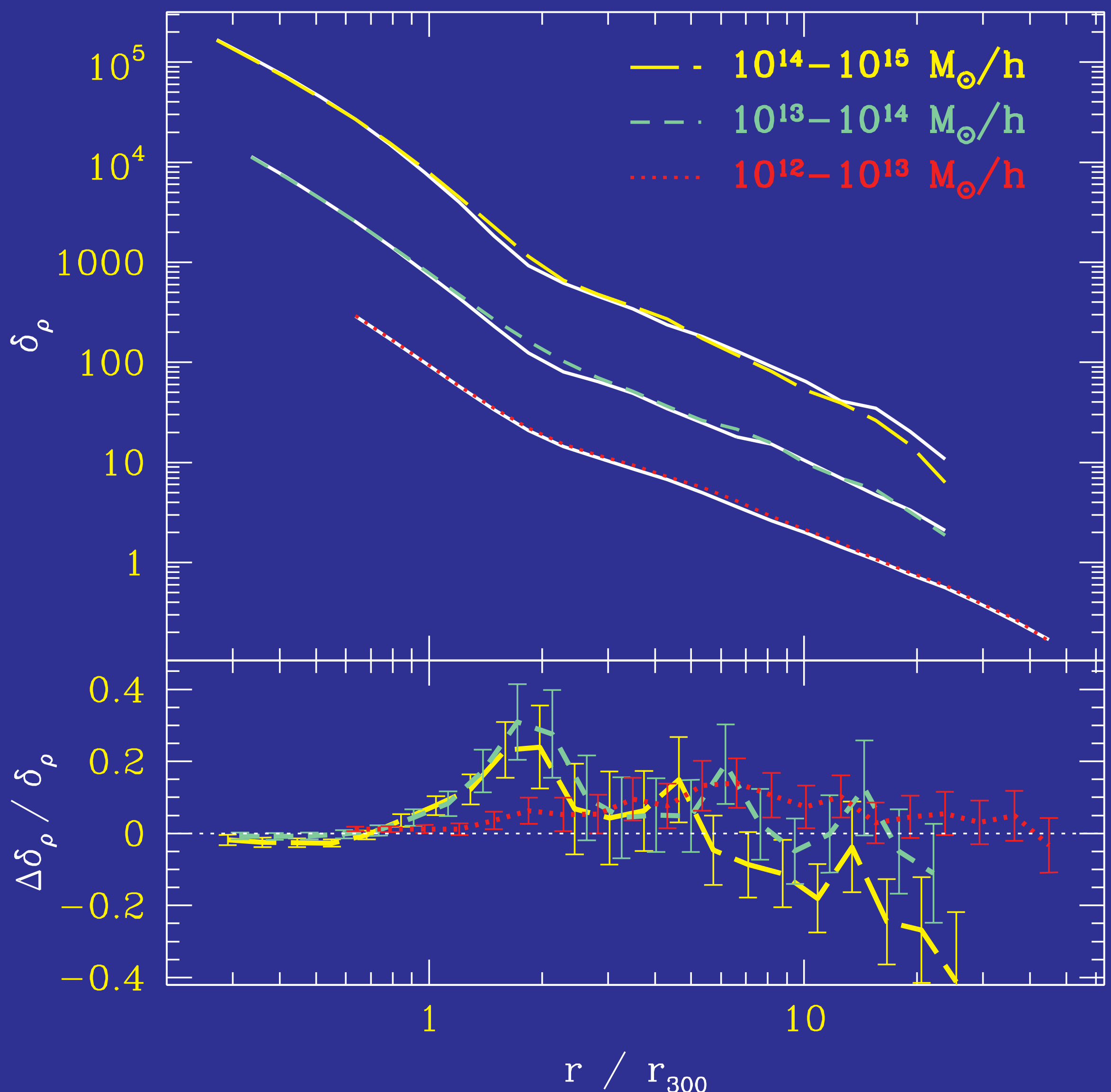
Halo Mass Correlation

- Enhanced forces vs lower bias



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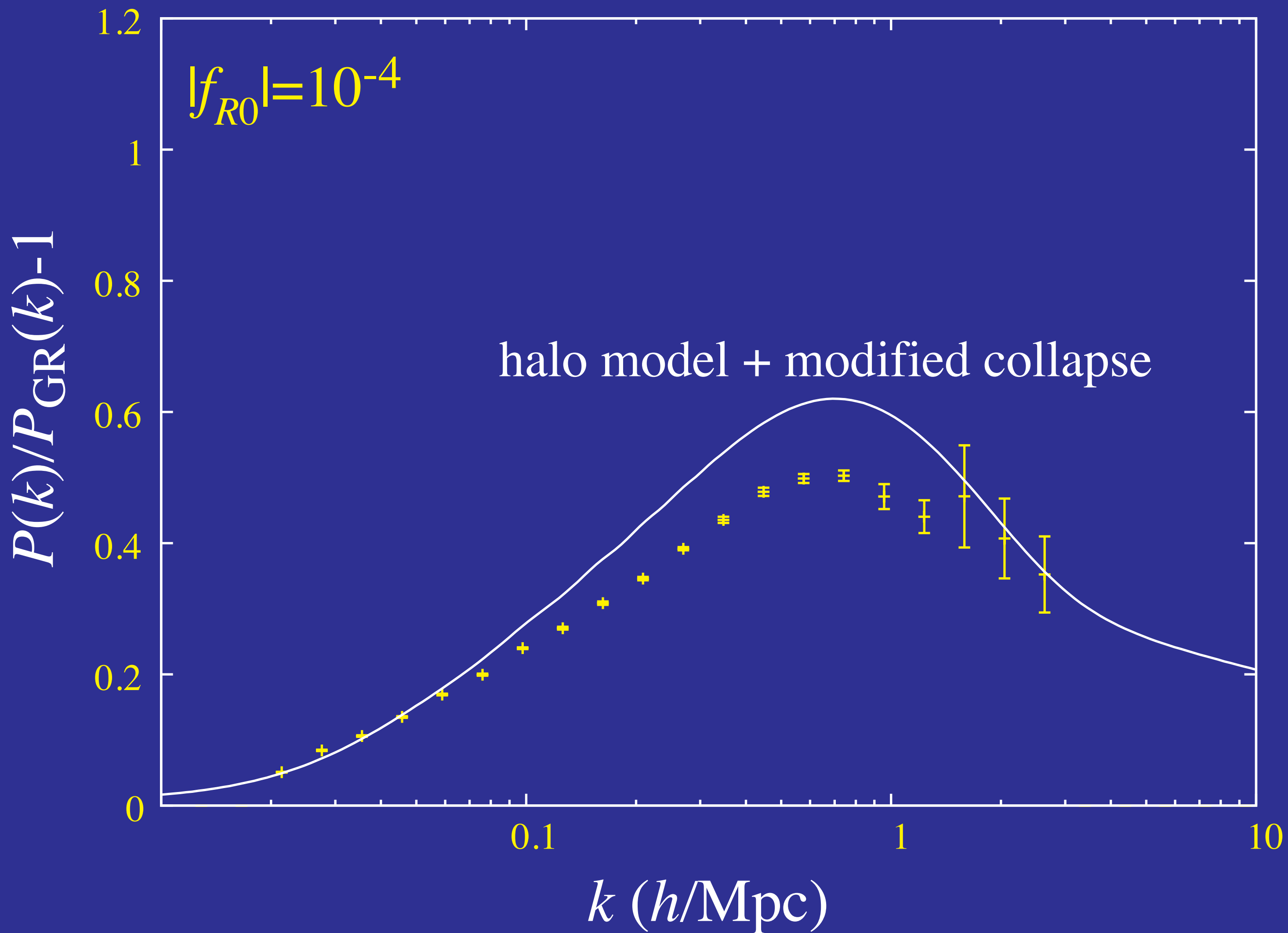
Halo Model

- Power spectrum trends also consistent with halos and modified collapse



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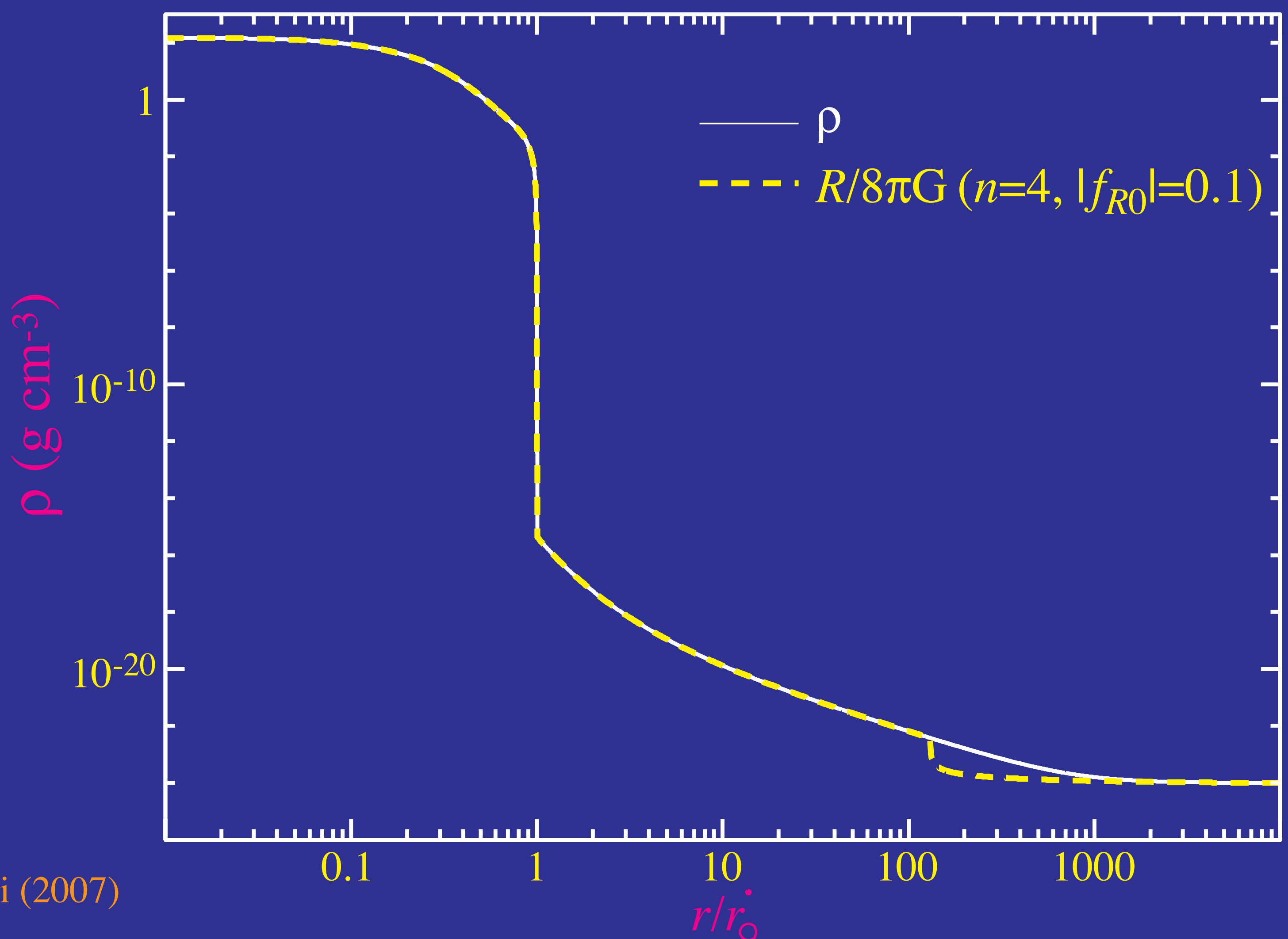
$f(R)$ Solar System Tests

Solar Profile

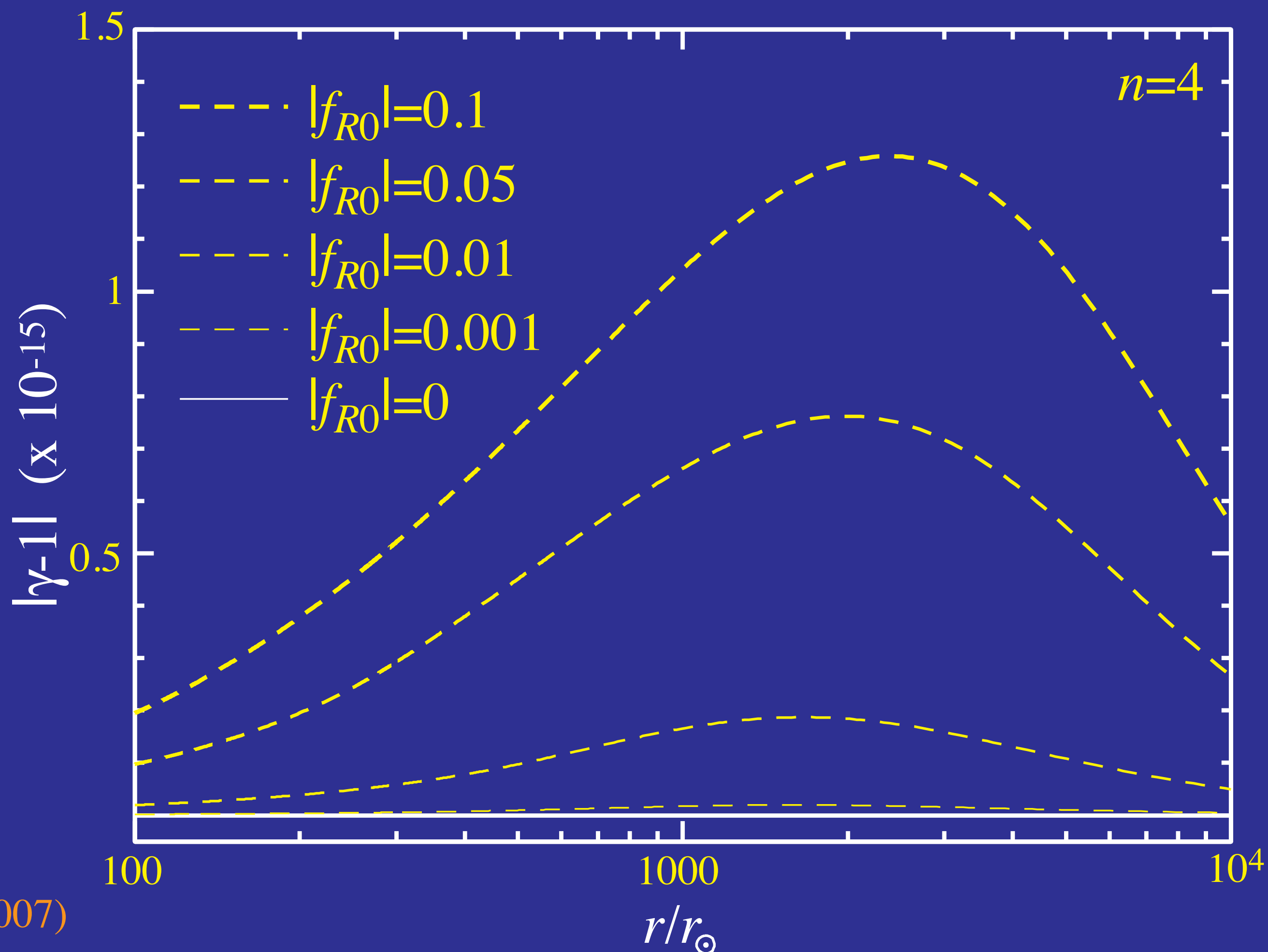
- Density profile of Sun is not a constant density sphere - interior photosphere, chromosphere, corona
- Density drops by ~25 orders of magnitude - does curvature follow?



Hu & Sawicki (2007)

Solar System Constraint

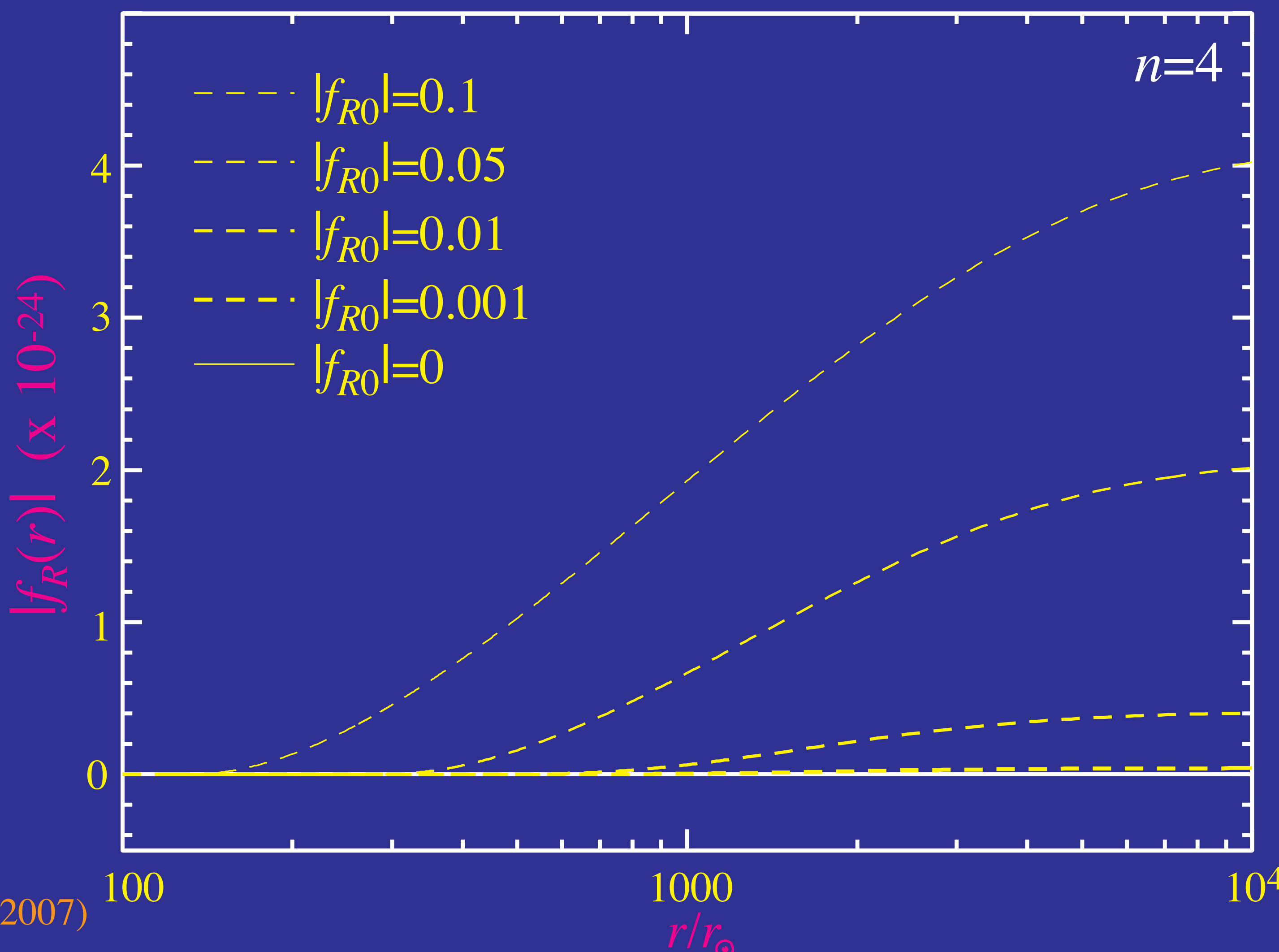
- Cassini constraint on PPN $|\gamma-1|<2.3\times10^{-5}$
- Easily satisfied if galactic field is at potential minimum $|f_{Rg}|<4.9\times10^{-11}$
- Allows even order unity cosmological fields



Hu & Sawicki (2007)

Field Solution

- Field solution smoothly relaxes from exterior value to high curvature interior value $f_R \sim 0$, minimizing potential + kinetic
- Juncture is where thin-shell criterion is satisfied $|\Delta f_R| \sim \Delta\Phi$



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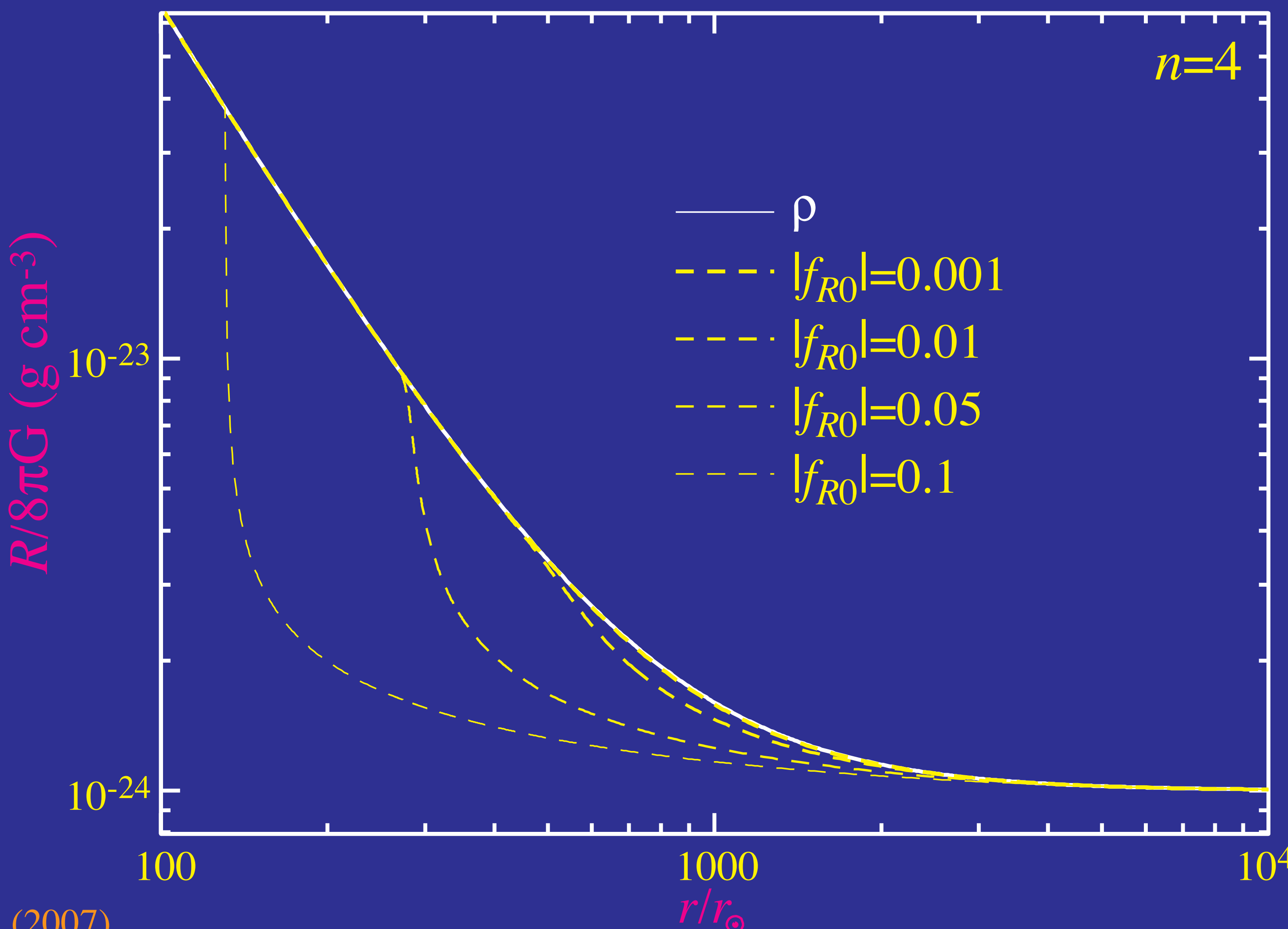
Solar Curvature

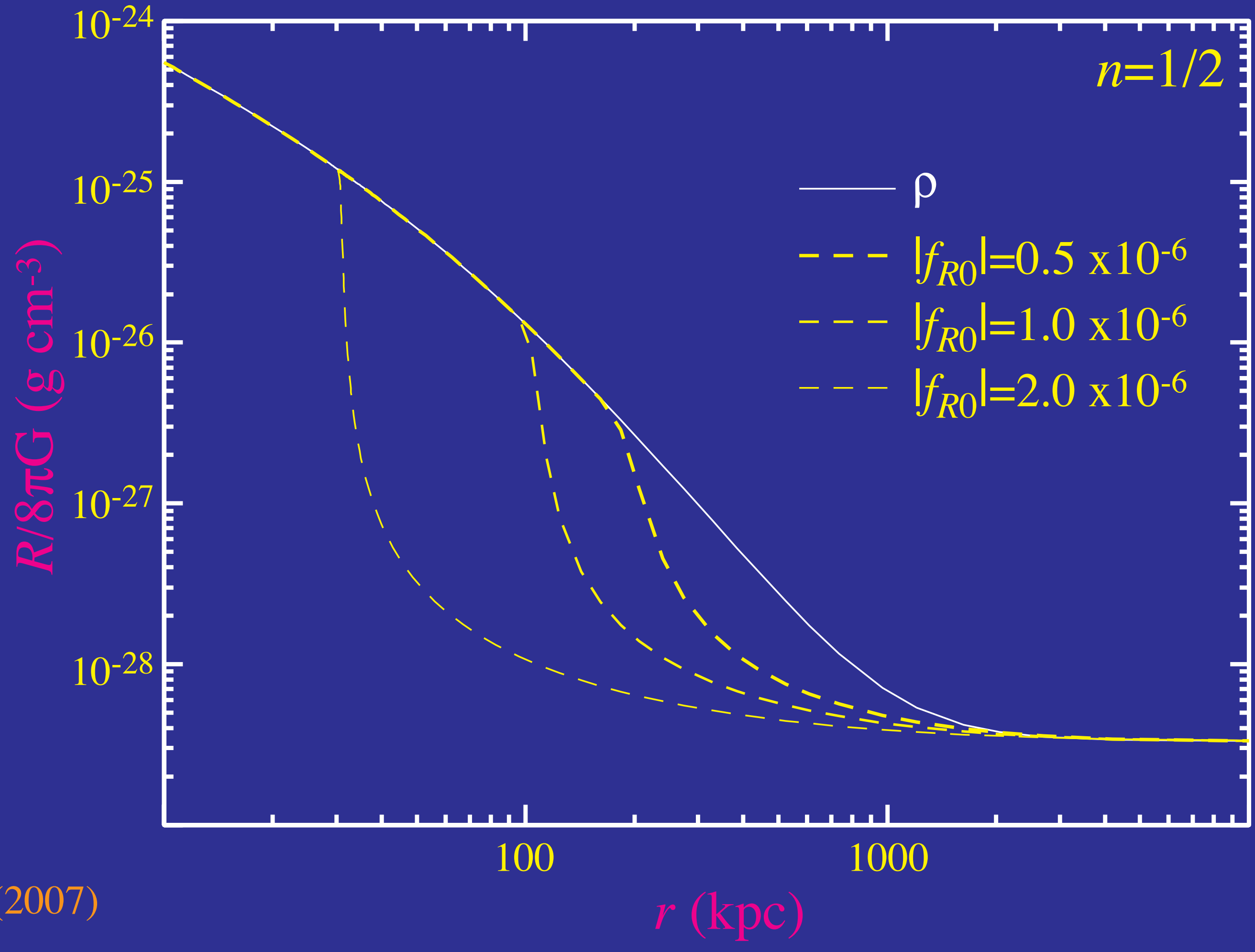
- Curvature drops suddenly as field moves slightly from zero
- Enters into low curvature regime where $R<8\pi G\rho$



Hu & Sawicki (2007)

Galactic Thin Shell

- Galaxy must have a thin shell for interior to remain at high curvature
- Rotation curve $v/c \sim 10^{-3}$, $\Phi \sim 10^{-6} \sim |\Delta f_R|$ limits cosmological field
- Has the low cosmological curvature propagated through local group and galactic exterior?



Hu & Sawicki (2007)

Summary

- General lessons from $f(R)$ example – 3 regimes:
  - large scales: conservation determined
  - intermediate scales: scalar-tensor
  - small scales: GR in high density regions, modified in low
- Given fixed expansion history $f(R)$ has additional continuous parameter: Compton wavelength
- Enhanced gravitational forces below environment-dependent Compton scale affect growth of structure
- Enhancement hidden by non-linear chameleon mechanism at high curvature $\neq$ high density)
- N-body (PM-relaxation) simulations show potentially observable differences in the power spectrum and mass function