Cosmological Perturbations: Isocurvature, Vorticity and Magnetic Fields

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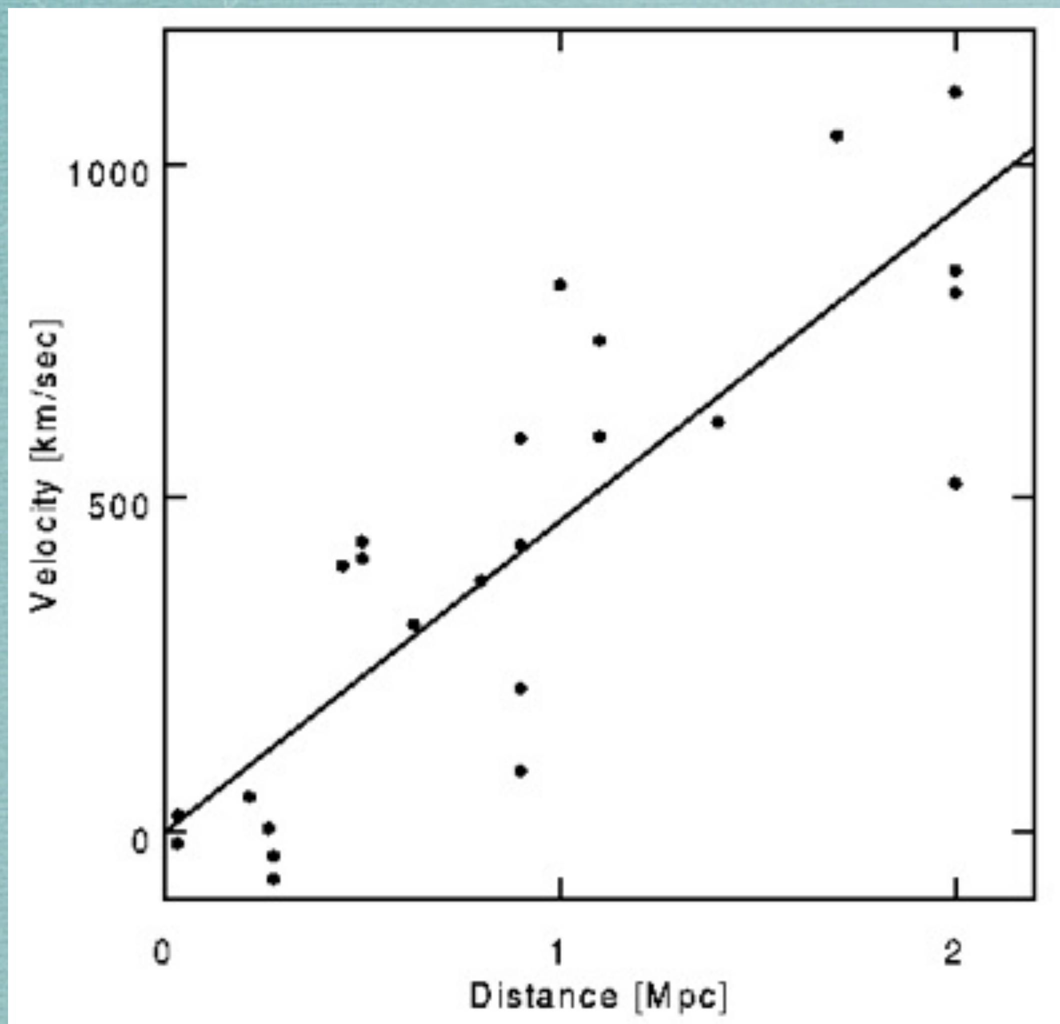
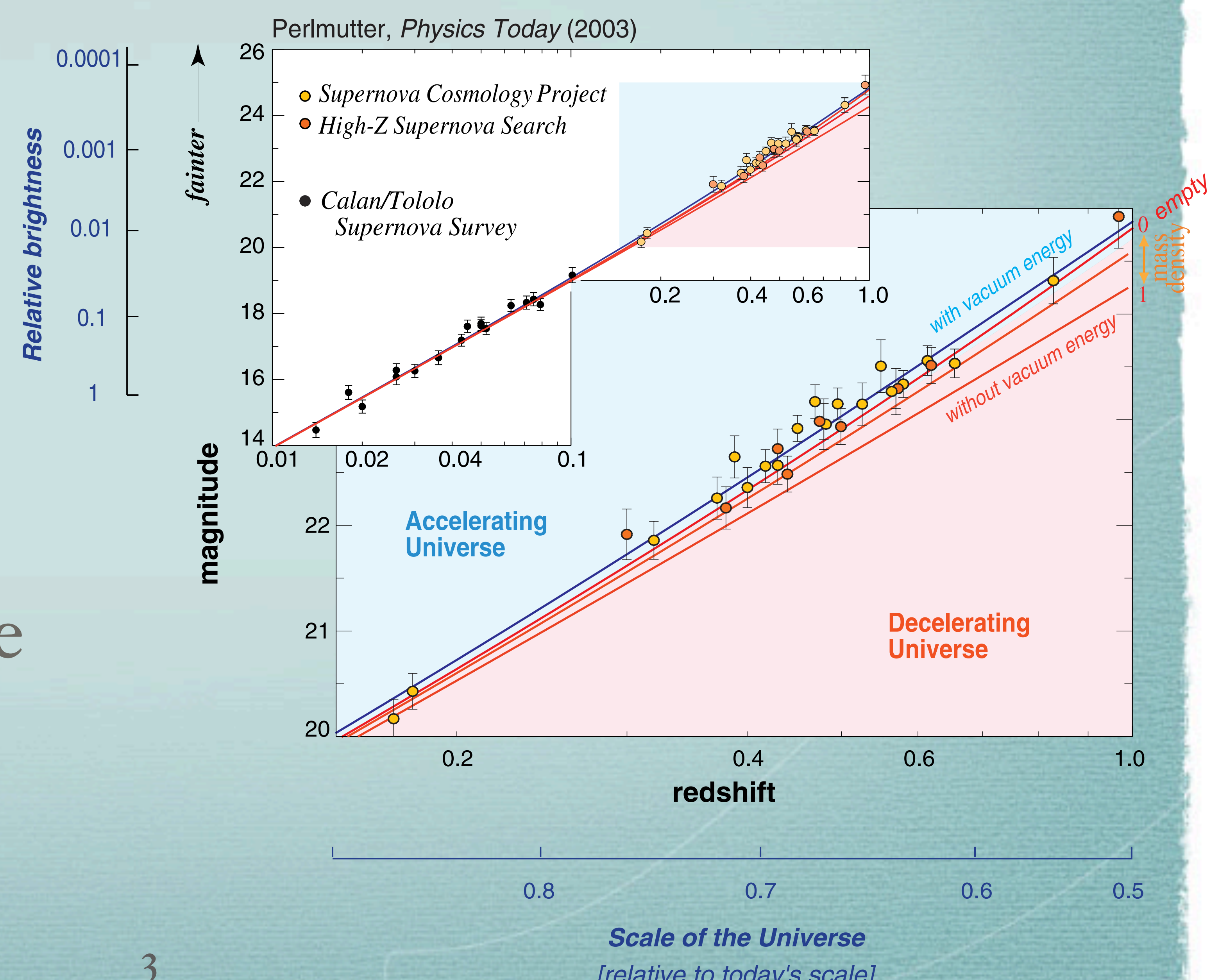
Outline

* Introduction and motivation
* Modelling inhomogeneities
* Vorticity in the early universe
* Isocurvature perturbations
  - in the concordance cosmology
  - in multi-field inflationary systems
* Observational significance of vorticity & magnetic fields

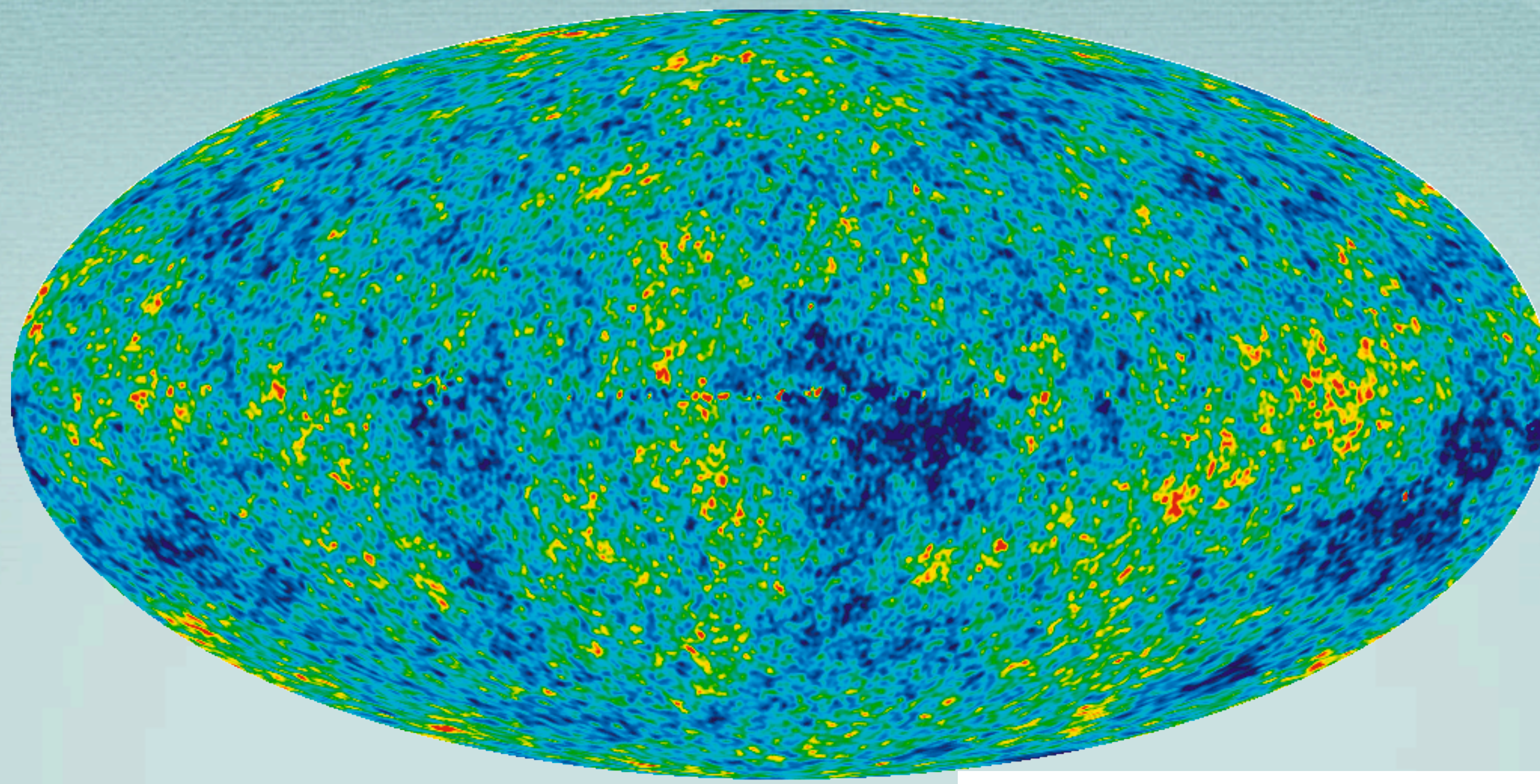
Observations

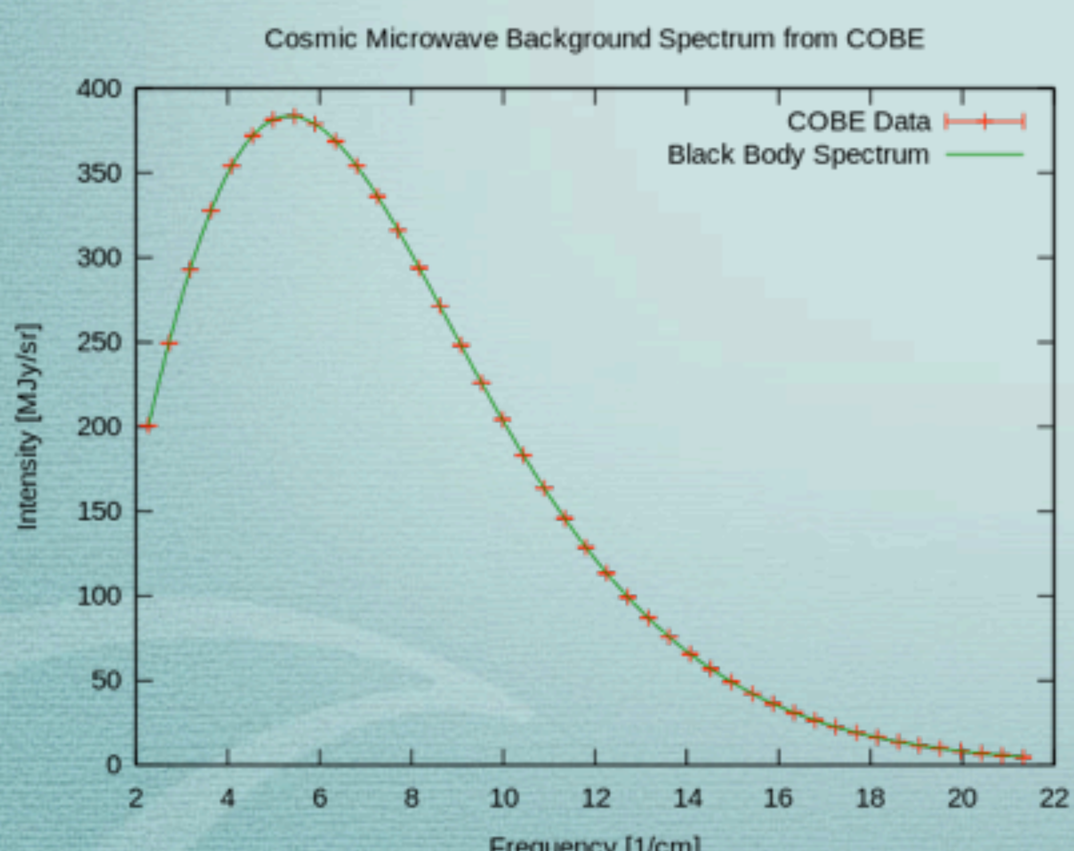


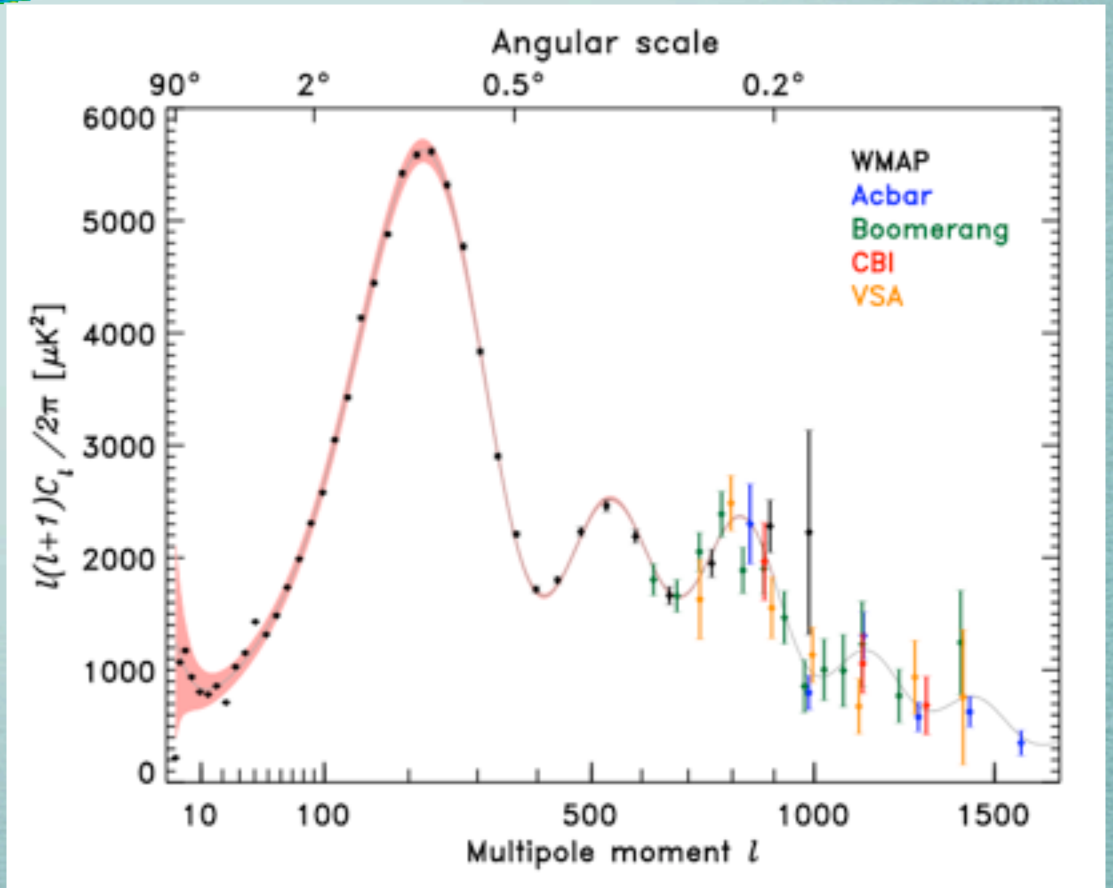


Hubble 1929 - first evidence for universe expansion

Microwave background







Modelling inhomogeneities

- Friedmann is an approximation: there exists structure (galaxies, stars, etc..), and CMB anisotropies
- Consider perturbations about a homogeneous 'background' solution
- e.g. write energy density as

$$\rho(\vec{x},t) = \bar{\rho}(t)\Big(1+\delta(\vec{x},t)\Big)$$

- newtonian mechanics...

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inhomogeneous perturbation

- newtonian mechanics...

Newtonian cosmology

* Newtonian perturbation theory:

energy density: $\rho(\vec{x},t) = \bar{\rho}(t)\Big(1+\delta(\vec{x},t)\Big)$

velocity: $\vec{v}(\vec{x},t)\,,$ Newtonian potential: $\Phi(\vec{x},t)$

* Fluid evolution equations

$$\dot{\delta} + \vec{\nabla}\cdot\Big[(1+\delta)\vec{v}\Big] = 0$$

$$\dot{\vec{v}} + H\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}\Phi - \frac{\vec{\nabla}P}{\bar{\rho}(1+\delta)}$$

* Poisson equation

$$\nabla^2\Phi = 4\pi G\bar{\rho}a^2\delta$$

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* Linearised fluid equations

$$\dot{\delta} + \vec{\nabla} \cdot \vec{v} = 0$$

$$\dot{\vec{v}} + H\vec{v} = -\vec{\nabla}\Phi - \frac{1}{\bar{\rho}}\vec{\nabla}\delta P$$

* Poisson equation

$$\nabla^2\Phi = 4\pi G\bar{\rho}a^2\delta$$

* Alternatively, writing $\delta P = c_{\rm s}^2 \delta\rho$, obtain

$$\frac{\partial^2 \delta}{\partial t^2} + 2H\frac{\partial \delta}{\partial t} = 4\pi G\bar{\rho}\delta + c_{\rm s}^2 \nabla^2 \delta$$

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Pressure term

Gravitational term: perturbations grow via gravitational instability

Relativistic inhomogeneities

* General relativity governs dynamics of the universe
* Must use relativity to describe regions of high density, fluids moving an appreciable fraction of c, or large scales
* Einstein's field equations:

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energy momentum tensor, describes matter

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Cosmological perturbations

* How to proceed?
  - Fully inhomogeneous solution (*extremely* difficult in principle; impossible in practice?)
  - Similar to Newtonian case: expand around a homogeneous solution - **Cosmological Perturbation Theory**
* Inhomogeneous perturbations to

matter, e.g., energy density $\rho(\vec{x},t) = \bar{\rho}(t)\Big(1 + \delta(\vec{x},t)\Big)$

geometry: metric tensor $g_{\mu\nu}(\vec{x},t) = g^{(0)}_{\mu\nu}(t) + \delta g_{\mu\nu}(\vec{x},t)$

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* FLRW metric:

$$[g^{(0)}_{\mu\nu}] = \begin{bmatrix} -1 & 0 \\ 0 & a^2(t)\delta_{ij} \end{bmatrix}$$

- homogeneous & isotropic
- take flat spatial space in agreement with observations

$$g_{\mu\nu}(\vec{x},t) = g^{(0)}_{\mu\nu}(t) + \delta g_{\mu\nu}(\vec{x},t)$$

* Perturbed FLRW metric:

two independent scalars, e.g.

$$[\delta g_{\mu\nu}] = \begin{bmatrix} -2\Phi(\vec{x},t) & 0 \\ 0 & a^2(t)2\Psi(\vec{x},t)\delta_{ij} \end{bmatrix}$$

or

$$[\delta g_{\mu\nu}] = \begin{bmatrix} -2\phi(\vec{x},t) & a(t)B_{,i}(\vec{x},t) \\ a(t)B_{,i}(\vec{x},t) & 0 \end{bmatrix}$$

Governing equations

- Fluid equations

$$\delta' + (1+w)(\nabla^2 v - 3\Psi') = 3\mathcal{H}(w - c_{\rm s}^2)\delta$$

$$v' + \mathcal{H}(1-3w)v + \frac{w'}{1+w}v + \frac{\delta P}{\bar\rho(1+w)} + \Phi = 0$$

- Poisson equation

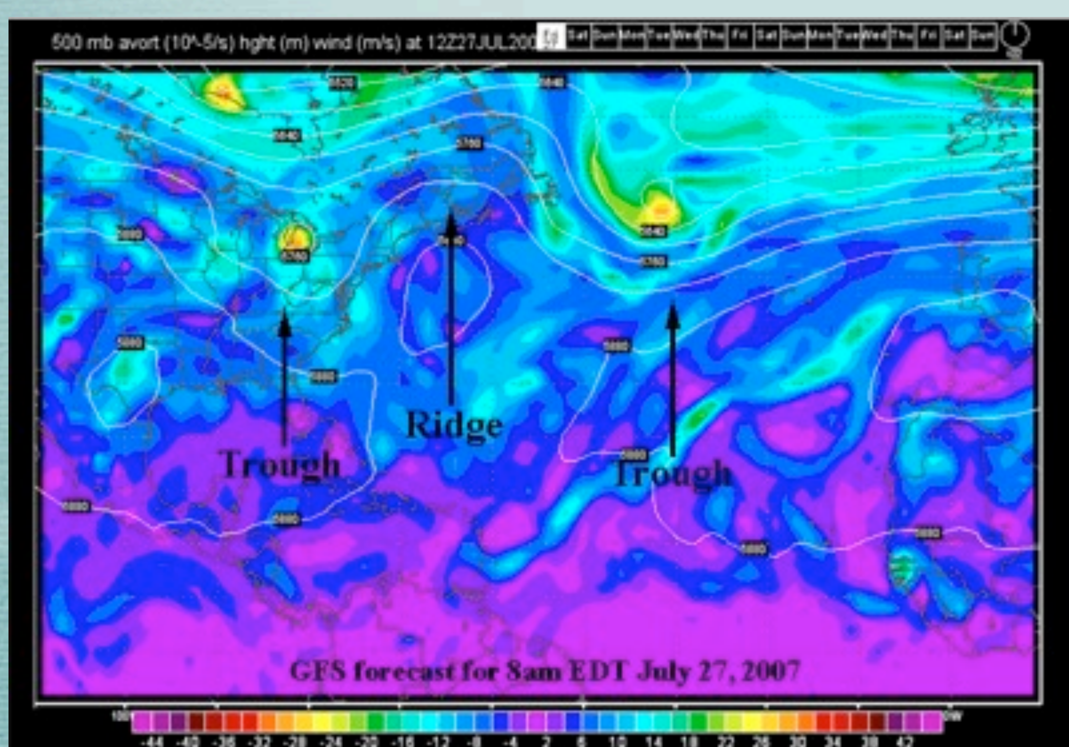
$$\nabla^2\Phi = -4\pi G a^2 \bar\rho\Big[\delta - 3\mathcal{H}(1+w)\nabla^2 v\Big]$$

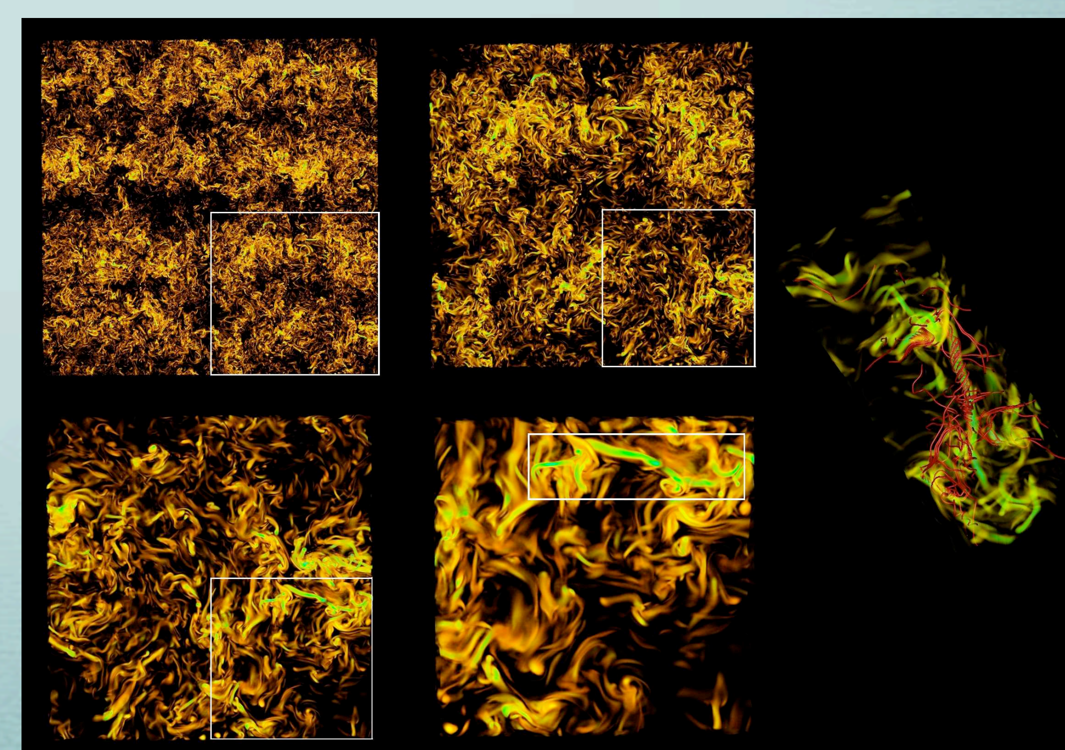
When Newtonian theory is not enough...

* But Newtonian theory cannot model
  - perturbations in relativistic species (radiation, neutrinos,...)
  - regions of high pressure (eg early universe)
  - regions of a comparable size of the horizon

* For the early universe (inflation/CMB) use relativistic pert thy
* Effects of general relativity on initial conditions for N-Body sims (in progress)

Vorticity





Classical fluids

* Classical fluid dynamics $\vec{\omega} \equiv \vec{\nabla} \times \vec{v}$
* Euler equation
$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla} P$$
* Evolution:
$$\frac{\partial \vec{\omega}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{\omega}) + \frac{1}{\rho^2} \vec{\nabla}\rho \times \vec{\nabla} P$$
  - 'source' term zero if $\vec{\nabla} P$ and $\vec{\nabla}\rho$ are parallel
  - i.e. barotropic fluid, no source term
* The inclusion of entropy provides a source for vorticity

Crocco (1937)

Entropy perturbations

* Adiabatic system $$\frac{\delta P}{\dot{P}} = \frac{\delta \rho}{\dot{\rho}}$$

* Non-adiabatic system allows for entropy perturbations

$$\frac{\delta P}{\dot{P}} \neq \frac{\delta \rho}{\dot{\rho}} \quad \Rightarrow \quad \delta P = \frac{\dot{P}}{\dot{\rho}}\delta\rho + \delta P_{\mathrm{nad}}$$

* Single fluid: require equation of state $P \equiv P(\rho, S)$

* Multiple fluids:
  - Intrinsic part of each fluid, $\delta P_{\alpha\mathrm{intr}} \equiv \delta P_\alpha - c_\alpha^2 \delta\rho_\alpha$
  - Relative part between fluids,

$$\delta P_{\mathrm{rel}} = \frac{1}{2\dot{\rho}} \sum_{\alpha,\beta} (c_\alpha^2 - c_\beta^2)(\dot{\rho}_\beta \delta\rho_\alpha - \dot{\rho}_\alpha \delta\rho_\beta)$$

Vorticity evolution

* Vorticity tensor, $\omega_{\mu\nu} = \mathcal{P}_\mu{}^\alpha \mathcal{P}_\nu{}^\beta u_{[\alpha;\beta]}$
* First order vorticity evolves as

$$\omega'_{1ij} - 3\mathcal{H}c_{\rm s}^2\omega_{1ij} = 0$$

Kodama & Sasaki (1984)

* Reproduces well known result that, in radiation domination,

$$|\omega_{1ij}\omega_1{}^{ij}| \propto a^{-2}$$

* i.e. in absence of anisotropic stress, no source term: $\omega_{1ij} = 0$ is a solution to the evolution equation

Beyond linear peturbations

* So far, have considered linear perturbations
* Extend, by expanding as, e.g.,

$$\rho(\vec{x},\eta) = \rho_0(\eta) + \delta\rho_1(\vec{x},\eta) + \frac{1}{2}\delta\rho_2(\vec{x},\eta)$$

* Crucial difference: scalar, vector and tensor perturbations no longer decouple

Vorticity evolution: second order

* Second order vorticity, ω_{2ij}, evolves as

$$\omega'_{2ij} - 3\mathcal{H}c_{\rm s}^2\omega_{2ij} = \frac{2a}{\rho_0 + P_0}\left\{3\mathcal{H}V_{1[i}\delta P_{{\rm nad}1,j]} + \frac{\delta\rho_{1,[j}\delta P_{{\rm nad}1,i]}}{\rho_0 + P_0}\right\}$$

assuming zero first order vorticity.

* For vanishing non-adiabatic pressure, vorticity decays as at first order

Lu et. al. (2009)

* Including entropy gives a non-zero source term

AJC, Malik & Matravers (2009)

* This generalises Crocco's theorem to an expanding framework

Isocurvature/entropy...

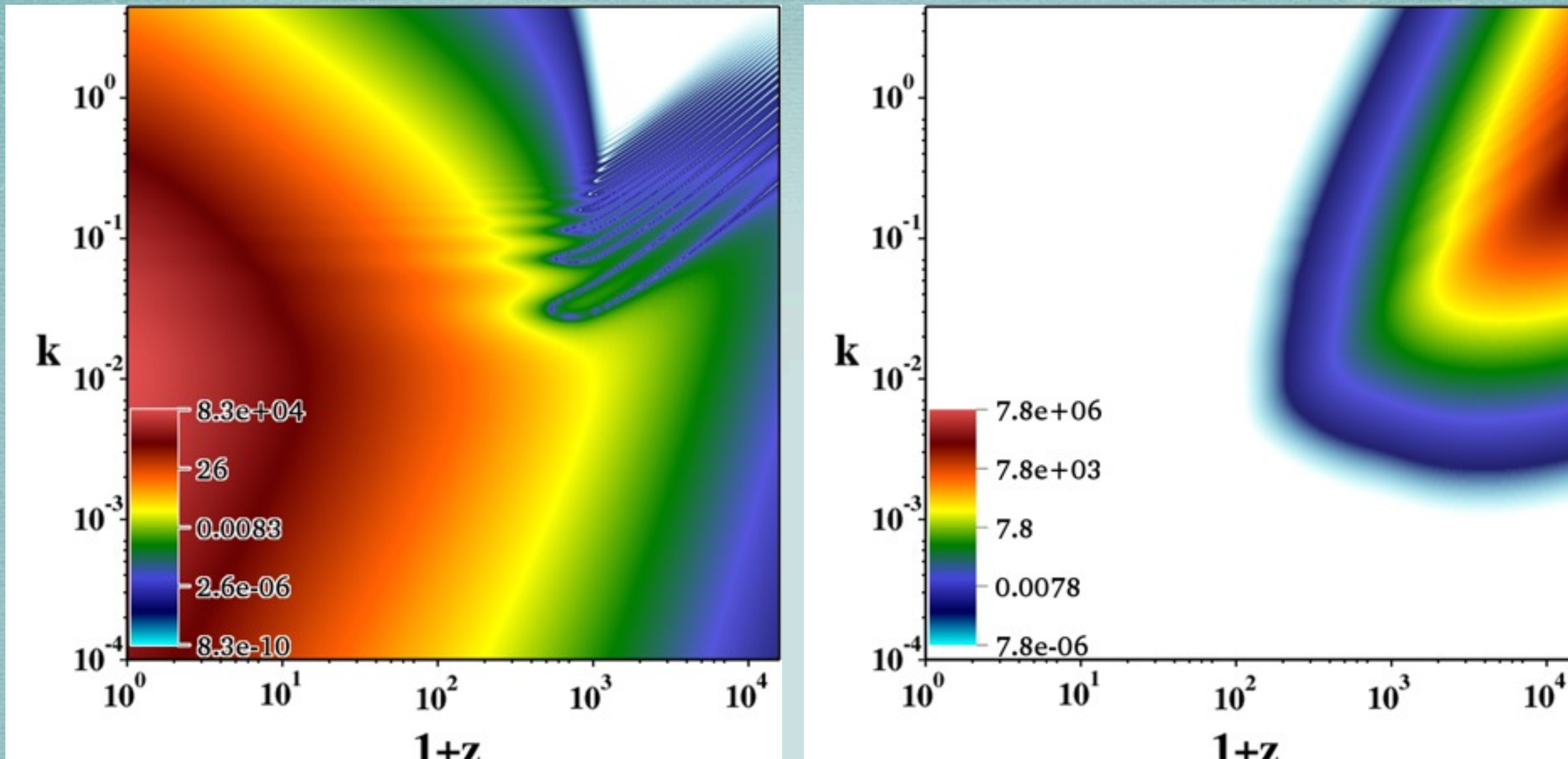
* Single (barotropic) fluid systems have zero non-adiabatic pressure
  - single scalar field, in superhorizon limit can be treated as a barotopic fluid
* Non-adiabatic pressure and entropy perturbations are gauge invariant, cannot be 'gauged away'
* Study:
  - relative entropy between fluids in the usual cosmic fluid (i.e. baryons, cold dark matter, radiation, neutrinos ...)
  - isocurvature perturbations in multi-field inflation model

... in concordance cosmology

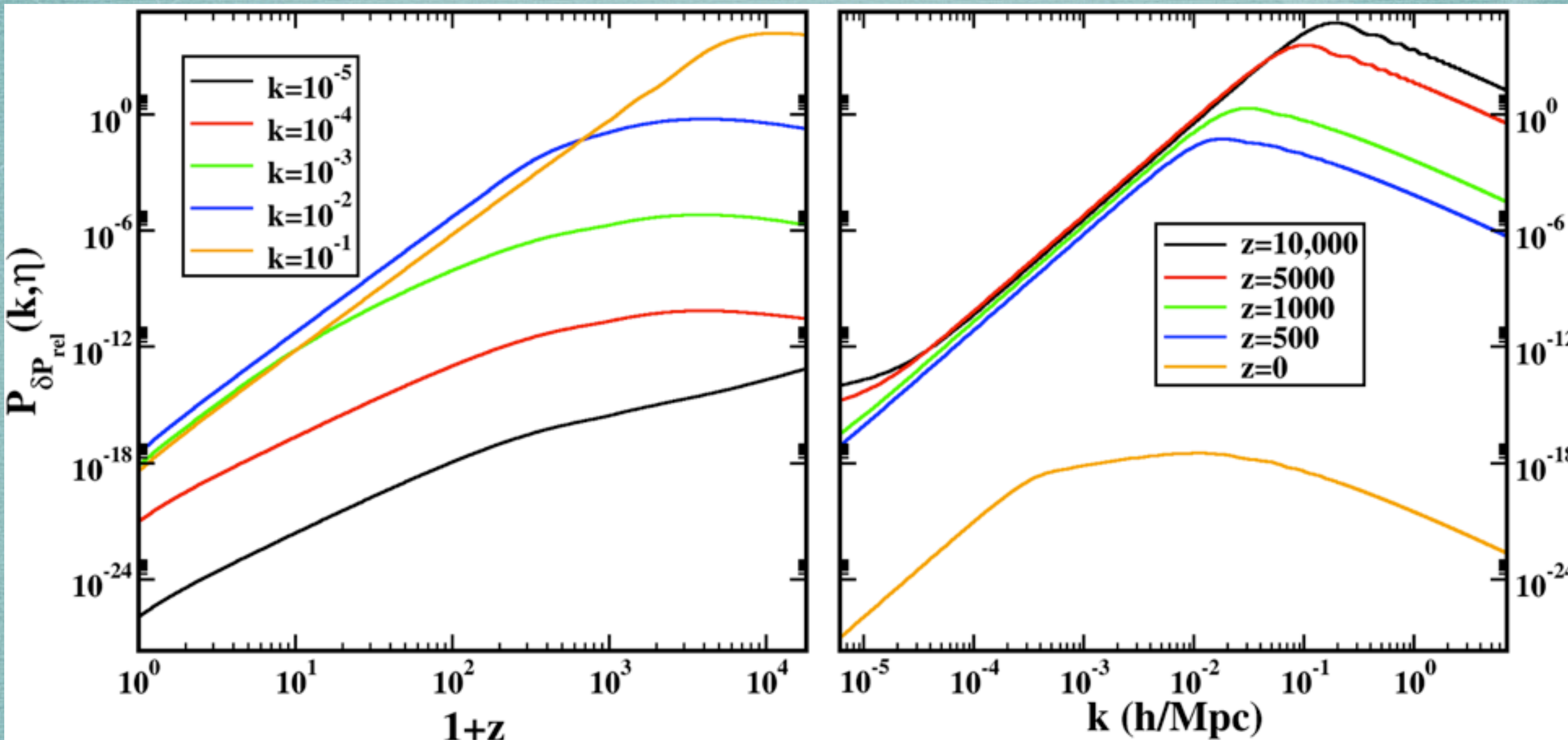
* baryons, CDM have $w_b = w_c = c_b^2 = c_c^2 = 0$
* photons, neutrinos are relativistic: $w_\gamma = w_\nu = c_\gamma^2 = c_\nu^2 = \frac{1}{3}$
* adiabatic initial conditions

$$\delta_\gamma = \delta_\nu = \frac{4}{3}\delta_b = \frac{4}{3}\delta_c = -\frac{2}{3}Ck^2\eta_i^2$$

* solve using a modified version of CMBFast
* aim: extract isocurvature already present in CMB calculations, to use as initial condition for vorticity



Power spectra of baryon density contrast $P_b(k, \eta)$ (left) and the non-adiabatic pressure perturbation $P_{\delta P_{\rm rel}}(k, \eta)$ (right)



$P_{\delta P_{\rm rel}}(k,\eta)$ as a function of redshift for set wavenumber (left); and as a function of wavenumber for set redshift (right).

Brown, AJC & Malik (2012)

... in multi-field inflation

* Consider two field inflation models with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\dot{\varphi}^2 + \dot{\chi}^2\right) + V(\varphi, \chi)$$

* Introduce comoving entropy perturbation

$$\mathcal{S} = \frac{H}{\dot{P}}\delta P_{\mathrm{nad}}$$

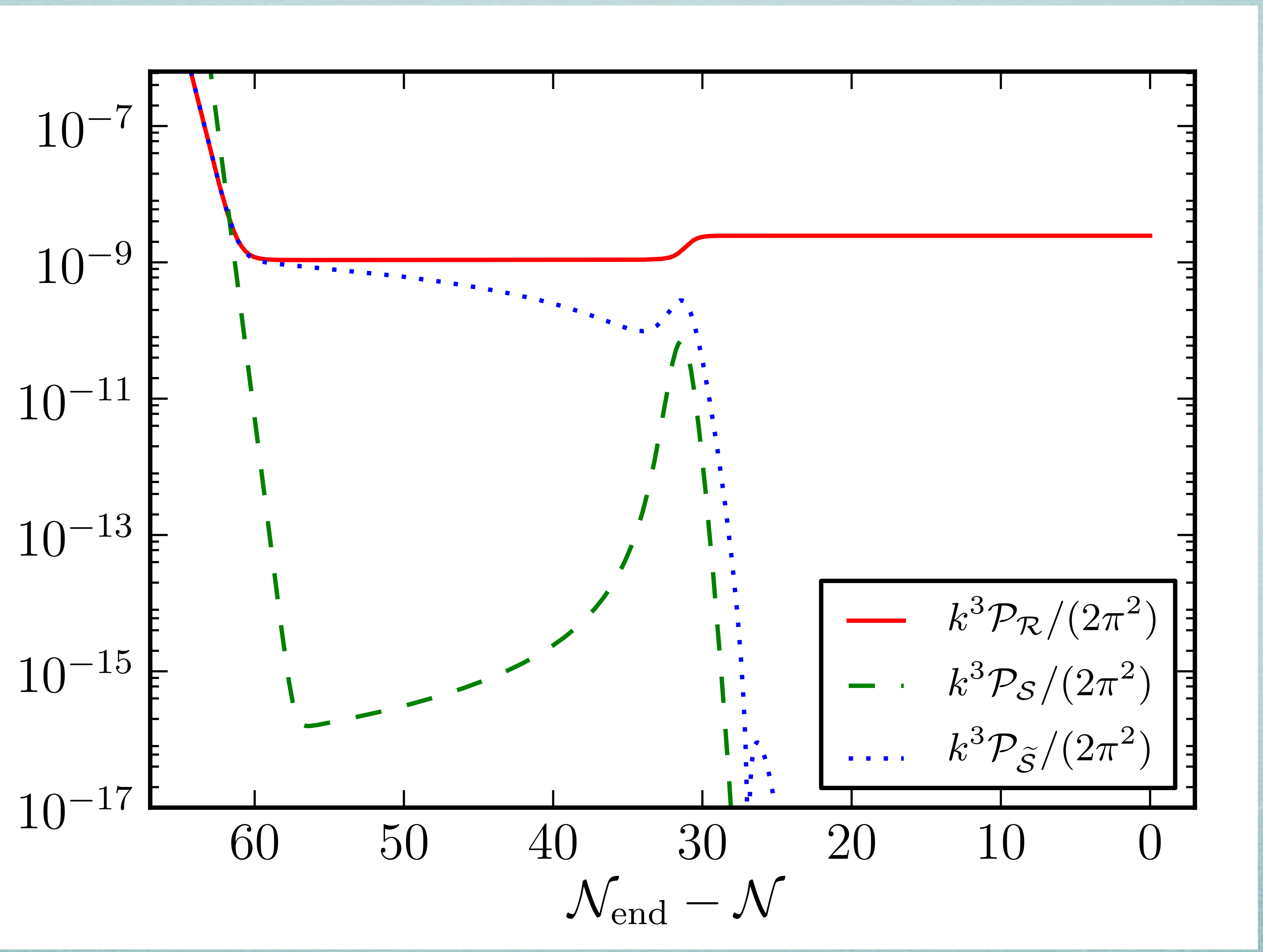
* To compare with comoving curvature perturbation

$$\mathcal{R} = \frac{H}{\dot{\varphi}^2 + \dot{\chi}^2}\left(\dot{\varphi}\delta\varphi + \dot{\chi}\delta\chi\right)$$

Then investigate dynamics of different models...

Double quadratic inflation

$$V(\varphi,\chi) = \frac{1}{2}m_\varphi^2\varphi^2 + \frac{1}{2}m_\chi^2\chi^2$$



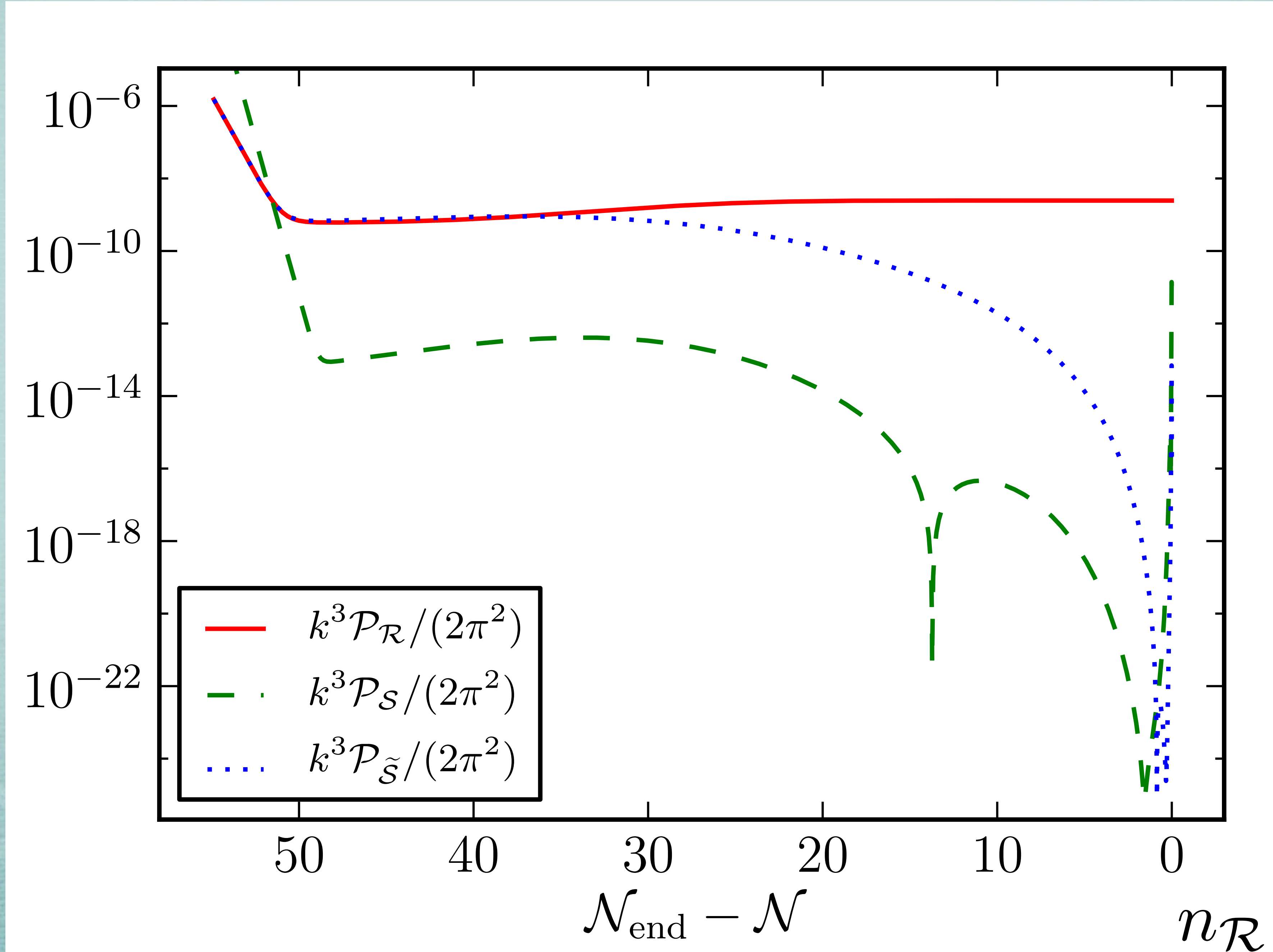
http://pyflation.ianhuston.net/

Huston & AJC (2012)

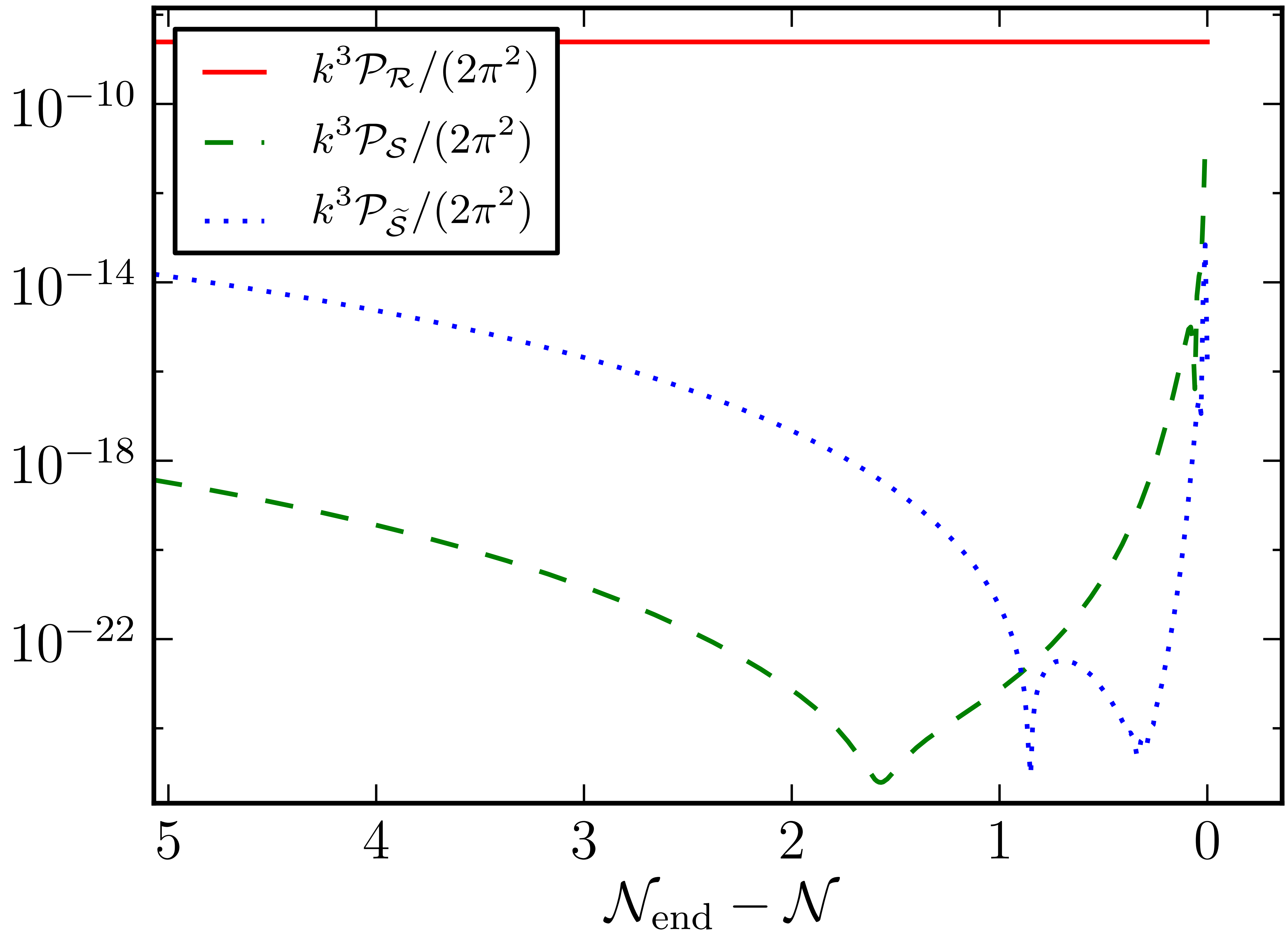
Double quartic inflation

$$V(\varphi,\chi) = \Lambda^4\left[\left(1-\frac{\chi_0^2}{v^2}\right)^2 + \frac{\varphi^2}{\mu^2} + \frac{2\varphi^2\chi^2}{\varphi_c^2 v^2}\right]$$

Avgoustidis et. al. (2011)



$n_{\mathcal{R}} = 0.932$


$k^3\mathcal{P}_{\mathcal{R}}/(2\pi^2)$
$k^3\mathcal{P}_{\mathcal{S}}/(2\pi^2)$
$k^3\mathcal{P}_{\tilde{\mathcal{S}}}/(2\pi^2)$
10^{-10}
10^{-14}
10^{-18}
10^{-22}
5
4
3
2
1
0
$\mathcal{N}_{\mathrm{end}} - \mathcal{N}$

Isocurvature: Summary

* Isocurvature is naturally sourced in concordance cosmology by relative entropy between species

* Two-field inflationary models can produce isocurvature at the end of inflation

* Future work:

  - Modelling reheating perturbatively with decay channels from fields to matter/radiation, how likely is isocurvature to survive?

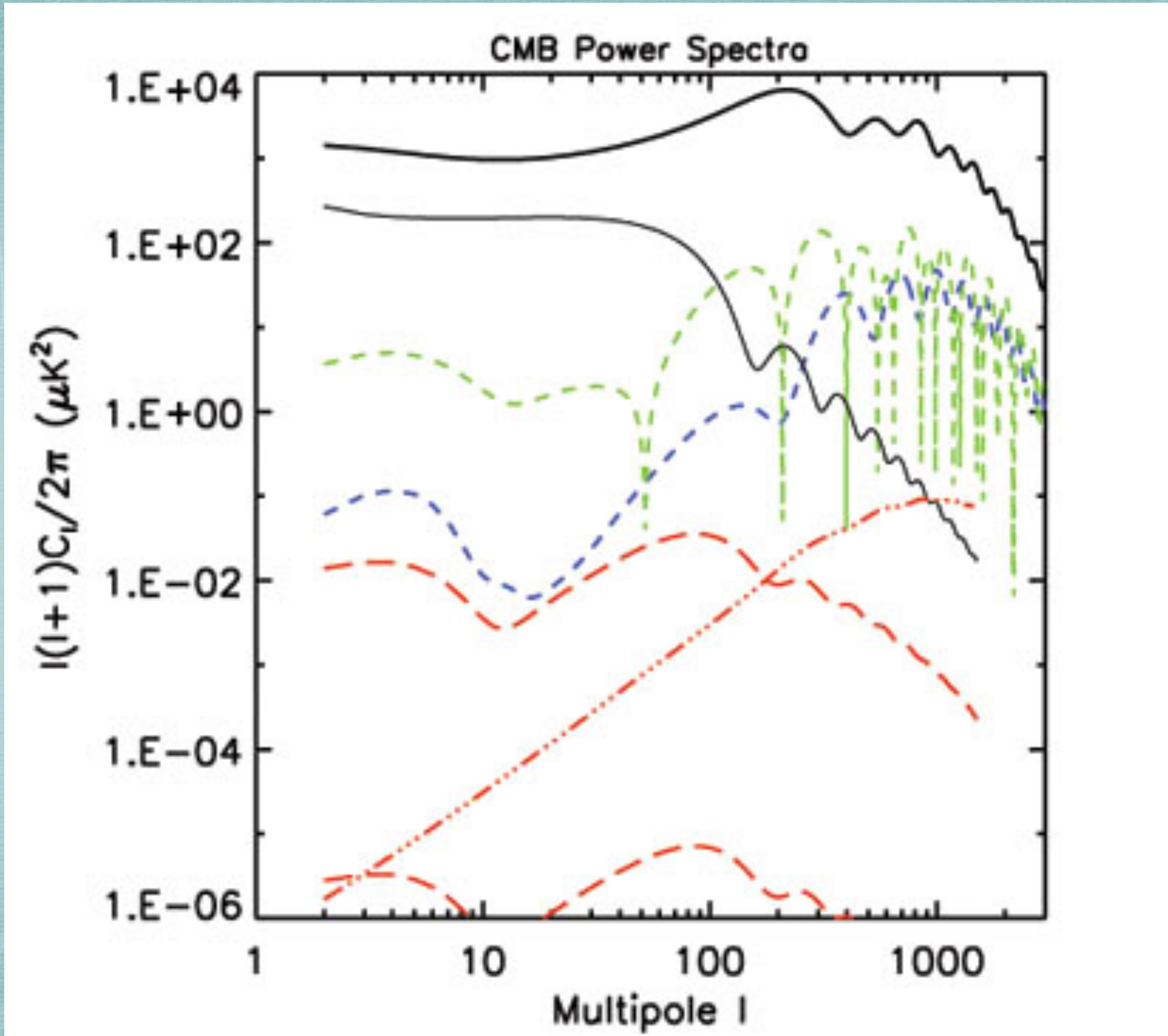
    Huston & AJC (in progress)

  - Can these realistic isocurvature perturbations generated a sizeable vorticity?

Observational Importance of Vorticity

Observational signatures

* For linear perturbations, B mode polarisation of the CMB only produced by tensor perturbations:
  - scalar perturbations only produce E mode polarisation
  - vectors produce B modes, but decay with expansion
* Second order, vector perturbations produced by first order density and entropy perturbations source B mode polarisation
* Important for current and future CMB polarisation expts


CMB Power Spectra
1.E+04
1.E+02
1.E+00
1.E-02
1.E-04
1.E-06
l(l+1)C$_l$/2π (μK^2)
1
10
100
1000
Multipole l

Magnetic Fields

- Electric and magnetic fields wrt observer u^μ

$$\boldsymbol{E}^\mu = F^{\mu\nu} u_\nu \qquad \boldsymbol{B}^\mu = \frac{1}{2}\epsilon^{\mu\nu\gamma\delta} u_\nu F_{\mu\delta}$$

- Governing equations are then Maxwell equations

$$F_{[\mu\nu;\lambda]} = 0 \qquad F^{\mu\nu}{}_{;\nu} = j^\mu$$

=> set of covariant Maxwell equations

- How to include in metric perturbation theory?
  - linear perturbations, include 'half-order, since $\boldsymbol{B}^2 \sim \rho$
  - unclear how to extend to higher order perturbation theory

* So, expand E/B fields using same expansion parameter

$$\boldsymbol{E}^\mu = \boldsymbol{E}_1^\mu + \frac{1}{2}\boldsymbol{E}_2^\mu + \frac{1}{6}\boldsymbol{E}_3^\mu \qquad \boldsymbol{B}^\mu = \boldsymbol{B}_1^\mu + \frac{1}{2}\boldsymbol{B}_2^\mu + \frac{1}{6}\boldsymbol{B}_3^\mu$$

* Maxwell equations at each order: e.g. 1st order

$$\partial_i {\boldsymbol{B}_1}^i = 0$$

$$\epsilon^{0ijk} a^2 \partial_j \boldsymbol{E}_{1k} = -{\boldsymbol{B}_1}^{i\prime} + \mathcal{H}{\boldsymbol{B}_1}^i\left(1 - \frac{2}{3}a\right)$$

$$\partial_i {\boldsymbol{E}_1}^i = -(j^\mu u_\mu)_1$$

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no source for linear magnetic field

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similarly at second order...

* Work to third order, interesting equation is:

$$\partial_i \boldsymbol{B}_3{}^i = -6\boldsymbol{E}_{1i}\omega_2{}^i$$

second order vorticity sourcing magnetic field!!

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$$
\begin{aligned}
&\frac{1}{6}\partial_i\mathcal{M}_3{}^i - \frac{1}{2}\mathcal{M}_2{}^{i\prime}(V_{1i} - B_{1i}) - \mathcal{M}_1{}^{i\prime}(\frac{1}{2}V_{2i} - \frac{1}{2}B_{2i} + V_{1i}\phi_1 - B_{1i}\phi_1)\\
&+\frac{1}{2}\mathcal{M}_1{}^j(2(2B_1{}^i - V_1{}^i)\partial_j B_{1i} + 14B_{1j}\mathcal{H}\phi_1 + 10\mathcal{H}\phi_1 V_{1j} - 8\phi_1\partial_j\phi_1 + 2\partial_j\phi_2 - 2V_1{}^i\partial_i V_{1j}\\
&+2B_1{}^i\partial_i V_{1j} - 7\mathcal{H}V_{2j} - 2V_{2j}{}' + 10B_{1j}\phi_1{}' + 4B_{1j}{}'\phi_1 + B_{2j}{}' - 8V_{1j}\phi_1{}' + 6\mathcal{H}B_{2j})\\
&+\mathcal{M}_2{}^i(3\mathcal{H}(B_{1i} - V_{1i}) + \frac{1}{2}(2\partial_i\phi_1 - \mathcal{H}V_{1i} - 2V_{1i}{}' + B_{1i}{}'))\\
&= -\mathcal{E}_{1i}\omega_2{}^i - \mathcal{E}_{2i}\omega_1{}^i + 2\mathcal{E}_{1i}V_1{}^i\omega_1{}^0 - 2\mathcal{E}_{1i}B_1{}^i\omega_1{}^0
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$$

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* Next - in progress (Ellie Nalson et al) - calculate size of this magnetic field generated; large enough to be seed field?
* Interesting formalism question: how does this compare to two-parameter perturbation theory?

Summary

* Vorticity generated at second order in perturbation theory from entropy perturbations
* Entropy perturbations can arise naturally in systems containing more than one fluid/field
  - cosmic fluid containing relativistic/non-relativistic matter
  - multi-field inflationary models
* This can source magnetic fields (albeit at third order)

Future directions

- Aim to calculate vorticity spectrum from realistic isocurvature inputs, such as:
  - inflation — Alabidi, AJC, Huston & White (in progress)
  - cosmic fluid — Brown, AJC & Malik (in progress)
- Investigate potential of second order vorticity to source primordial magnetic seed fields

  Nalson, AJC, Malik (in progress)
- Study effects of second order vorticity on B-mode polarisation